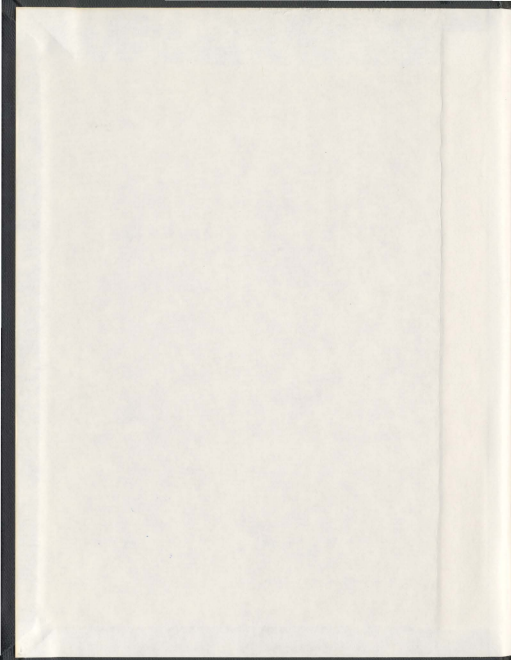


INFERENCES IN VOLATILITY MODELS

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Inferences in Volatility Models

by

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Abstract

In some real life time series, especially in financial time series, the variance of the responses over time appear to be non-stationary. The changes in the variances of such data are usually modeled through a dynamic relationship among these variances, and subsequently the responses are modeled in terms of the non-stationary variances. This type of time series model is referred to as the stochastic volatility model. However, obtaining the consistent and efficient estimators for the parameters of such a model has been proven to be difficult. Among the existing estimation approaches, the so-called generalized method of moments (GMM) and the quasi-maximum likelihood (QML) estimation techniques are widely used. In this thesis, we introduce a simpler method of moments (SMM), which, unlike the existing GMM approach, does not require an arbitrarily large number of unbiased moment functions to construct moment estimating equations for the parameters involved. We also demonstrate numerically that the proposed SMM approach is asymptotically more efficient than the existing QML approach. We also provide another simpler 'working' generalized quasi likelihood

(WGQL) approach which is similar but different than the SMM approach. Furthermore, the small and large sample behavior of the SMM and WGQL approaches are examined through a simulation study. The effect of the SMM estimation approach is also examined for kurtosis estimation.

In volatility models mentioned above, the responses are assumed to be uncorrelated. However, in some situations, it may happen that the responses become influenced by certain time dependent covariates, and as opposed to the standard stochastic volatility models, the responses become correlated. In the later part of thesis, we introduce an observation-driven dynamic (ODD) regression model with non-stationary error variances, these variances being modeled as in the standard stochastic volatility models. We refer to such a model as the observation driven dynamic-dynamic (dynamic²) (ODDD) volatility model. The parameters of this wider model are estimated by using a hybrid estimation technique by combining the GQL and SMM approaches.

Dedication

*Dedicated to my late father Saravanamuthu Pathmanathan,
my mother Nageswary Pathmanathan and my husband Mariathas
Judes Tagore.*

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Chapter 1

Introduction

The analysis of Gaussian time series data with non-stationary means and a suitable correlation structure has a long history both in statistics and econometrics literature. See for example, Box-Jenkins (1994) and Harvey (1989). In a financial time series analysis, it was, however, observed by (Black and Scholes (1972, p.416)) that the variances in stock returns and/or exchange rate data may change over various time intervals. In their concluding remarks, these authors, therefore, emphasized on research in this direction. Note that the modelling of this type of non-stationary variances based time series data is done either by using a suitable dynamic relationship for the variance at a given interval with the variances from the past time intervals; or by using a suitable dynamic relationship for the variance at a given time interval with the past squared observations. The former model is generally referred to

as the stochastic volatility (SV) and the later model is known as the autoregressive conditional heteroscedastic (ARCH) model. When the variance at a given time point satisfies a relationship with past variances as well as past squared observations, the model is referred to as the GARCH (Generalized ARCH) model. Since the pioneering work of the nobel laureate Engle (1982) [see also Engle and Kraft (1983), Engle et.al (1985), Engle and Bollerslev (1986), Bollerslev et.al (1992), Engle and Kroner (1995), Engle (2004)], the aforementioned models have been widely applied in the econometric literature. For example, we refer to Taylor (1982), Anderson and Sorensen (1996), Harvey et.al (1994), Ruiz (1994), Taylor (1994), Durbin and Koopman (2000), Broto and Ruiz (2004) Bollerslev (1986), Engle and Gonzalez-Rivera (1991) for the application of SV, ARCH and GARCH models.

As far as inferences in the SV, ARCH and GARCH models are concerned, the aforementioned papers including the papers by Bodurtha and Mark (1991), Simon (1989) use the so-called generalized method of moments (GMM) and/or quasi-maximumlikelihood (QML) approaches. These approaches are however numerically cumbersome and also they may not be efficient as compared to certain simpler approaches. In Chapter 2, we provide some simpler and efficient approaches for the estimation of the parameters of the SV model. However, before developing the new approaches, we, for convenience, explain the existing inferences such as GMM and QML techniques in Section 1.2.1 and 1.2.2 respectively. In Section 1.1, we present the existing SV model

along with its basic properties. The ARCH model is also discussed in brief in the same section.

1.1 Existing Volatility Models

1.1.1 Stochastic Volatility (SV) Model

Let y_t be the response at time t ($t = 1, \dots, T$). Suppose that σ_t^2 denotes a random and unobserved variance of y_t . If σ_t^2 were, however, known, then it is referred to as the conditional variance of y_t , i.e. $\text{Var}(Y_t|\sigma_t) = \sigma_t^2$. Note that the conditional Gaussian time series data $\{y_t\}$ with mean zero and variance $\{\sigma_t^2\}$ may be modelled as

$$y_t = \sigma_t \epsilon_t \quad t = 1, \dots, T, \quad (1.1)$$

[Taylor (1982), for example] where error variables ϵ_t 's are independently and identically (iid) normally distributed with mean zero and variance 1, that is, $\epsilon_t \stackrel{iid}{\sim} N(0, 1)$.

Also, ϵ_t and σ_t are assumed to be independent. Since σ_t^2 is unknown and it is reasonable to assume that the correlations between σ_t^2 's may decay as time lag increases, Taylor (1982), Anderson and Sorensen (1996) among others, have used Gaussian AR(1) type process to model the variances. That is,

$$\ln(\sigma_t^2) = h_t = \gamma_0 + \gamma_1 h_{t-1} + \eta_t; \quad t = 2, \dots, T, \quad (1.2)$$

where γ_0 is the intercept parameter, γ_1 is the volatility persistence parameter and $\eta_t \stackrel{iid}{\sim} N(0, \sigma_\eta^2)$ with σ_η^2 as the measure of uncertainty about future volatility. Note

that if $|\gamma_1| < 1$, then h_t 's follow a stationary AR(1) process. In (1.2), similar to Lee and Koopman (2004, eqn (1.1c)) it is reasonable to assume that

$$\ln(\sigma_t^2) = h_t \stackrel{iid}{\sim} N\left(\frac{\gamma_0}{1-\gamma_1}, \frac{\sigma_\eta^2}{1-\gamma_1^2}\right). \quad (1.3)$$

Without any loss of generality we will use this stationarity assumptions all through out in the thesis.

1.1.1.1 Basic Properties of Stochastic Volatility Model:

(a) Asymptotic Mean, Variance and Covariance in $\{Y_t\}$

Since h_t follows the AR(1) process (1.2), it may be shown that the unconditional mean of y_t is zero by (1.1). That is,

$$E[Y_t] = E_{\sigma_t} E[Y_t | \sigma_t] = E[\sigma_t E[\epsilon_t]] = 0, \quad (1.4)$$

as σ_t^2 and ϵ_t are independent. Next, the unconditional variance of y_t , i.e., $\text{Var}(Y_t)$, we write

$$\begin{aligned} \text{Var}(Y_t) &= E_{\sigma_t^2} [\text{Var}(Y_t | \sigma_t^2)] + \text{Var}_{\sigma_t^2} [E(Y_t | \sigma_t^2)] \\ &= E_{\sigma_t^2} [\sigma_t^2 \text{Var}(\epsilon_t)] = E[\sigma_t^2]. \end{aligned} \quad (1.5)$$

Since h_t has the AR(1) relationship as in (1.2), it is clear that for $|\gamma_1| < 1$, $h_t = \log \sigma_t^2$ has the asymptotic mean and variance given by

$$\lim_{t \rightarrow \infty} E[h_t] = \frac{\gamma_0}{1-\gamma_1} = \mu_h \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{Var}[h_t] = \frac{\sigma_\eta^2}{1-\gamma_1^2} = \sigma_h^2, \quad (1.6)$$

respectively [see also Harvey, 1994, p.249, Jacquier, 1994, p.386 and Anderson and Sorenson, 1996, p.331]. Now, as $\sigma_t^2 = \exp(h_t)$, where h_t follows the normal distribution by (1.2), by using the moment generating function of h_t , we obtain the asymptotic variance of y_t as

$$\lim_{t \rightarrow \infty} \text{Var}(Y_t) = \lim_{t \rightarrow \infty} E[\sigma_t^2] = \exp(\mu_h + \frac{\sigma_h^2}{2}) = \bar{\sigma}^2 (\text{say}). \quad (1.7)$$

[see also Tsay, 2005, p.134, Mills, 1999, p.127, Jacquier, 1994, p.386, and Anderson and Sorenson, 1996, p.331]. Next, by using the definition of the covariance, the lag k ($k=1, \dots, t-1$) unconditional covariance between Y_{t-k} and Y_t may be computed as

$$\begin{aligned} \text{Cov}(Y_{t-k}, Y_t) &= E[(Y_{t-k} - E[Y_{t-k}])(Y_t - E[Y_t])] = E[Y_{t-k}Y_t] - E[Y_{t-k}]E[Y_t] \\ &= E[\sigma_{t-k}\sigma_t\epsilon_{t-k}\epsilon_t] = E[\sigma_{t-k}\sigma_t]E[\epsilon_{t-k}\epsilon_t] = 0, \end{aligned} \quad (1.8)$$

as $E[Y_t] = 0$; σ_t and ϵ_t are independent; and also $\epsilon_t \stackrel{iid}{\sim} N(0, 1)$.

(b) Asymptotic Mean, Variance and Covariance in $\{Y_t^2\}$.

Note that the mean, variance and covariance of $\{y_t\}$ are given by (1.4), (1.5) and (1.8), respectively. However, as one is interested to fit the SV model (1.1) - (1.2) to the data, it is important to estimate the parameters γ_0 , γ_1 and σ_t^2 involved in (1.2). Consequently, it is natural that these parameters be estimated by using $\{Y_t^2\}$ rather than $\{Y_t\}$. The unconditional mean of Y_t^2 is

$$\lim_{t \rightarrow \infty} E[Y_t^2] = \lim_{t \rightarrow \infty} \text{Var}[Y_t] = \exp(\mu_h + \frac{\sigma_h^2}{2}) = \bar{\sigma}^2, \quad (1.9)$$

which is given by (1.7). Next, because $E[\epsilon_t^4] = 3$, in the fashion similar to that of (1.7), we compute the limiting variance of Y_t^2 by using

$$\begin{aligned}\lim_{t \rightarrow \infty} \text{Var}[Y_t^2] &= \lim_{t \rightarrow \infty} [E[Y_t^4] - (E[Y_t^2])^2] = \lim_{t \rightarrow \infty} E_{\sigma_t^2} E[\sigma_t^4 \epsilon_t^4 | \sigma_t^2] - \bar{\sigma}^4 = 3 \lim_{t \rightarrow \infty} E[\sigma_t^4] - \bar{\sigma}^4 \\ &= 3 \exp[2\mu_h + 2\sigma_h^2] - \bar{\sigma}^4 \\ &= \exp[2\mu_h + \sigma_h^2] [3 \exp(\sigma_h^2) - 1],\end{aligned}\quad (1.10)$$

(see also Tsay, 2005, p.134, Mills, 1999, p.128, Jacquier, 1994, p.386 and Anderson and Sorenson, 1996, p.331). Next, by definition, the limiting lag k ($k=1, \dots, t-1$) covariance between the squared responses Y_{t-k}^2 and Y_t^2 is given by

$$\begin{aligned}\lim_{t \rightarrow \infty} \text{Cov}(Y_{t-k}^2, Y_t^2) &= \lim_{t \rightarrow \infty} [E[(Y_{t-k}^2 - E[Y_{t-k}^2])(Y_t^2 - E[Y_t^2])] = \lim_{t \rightarrow \infty} E[Y_{t-k}^2 Y_t^2] - \bar{\sigma}^4 \\ &= \lim_{t \rightarrow \infty} E[\sigma_{t-k}^2 \sigma_t^2 \epsilon_{t-k}^2 \epsilon_t^2] - \bar{\sigma}^4 = \lim_{t \rightarrow \infty} E[\sigma_{t-k}^2 \sigma_t^2] - \bar{\sigma}^4.\end{aligned}\quad (1.11)$$

Next, by using the dynamic relationship (1.2) and the moment generating function, we obtain,

$$\lim_{t \rightarrow \infty} E[\sigma_{t-k}^2 \sigma_t^2] = \exp\left[2\frac{\gamma_0}{1-\gamma_1} + \left(\frac{\sigma_h^2}{1-\gamma_1^2}(1+\gamma_1^k)\right)\right] = \exp\left[2\mu_h + \sigma_h^2(1+\gamma_1^k)\right],\quad (1.12)$$

(see also Mills, 1999, p.128, Anderson and Sorenson 1996, p.331 and Jacquier et.al, 1994, p. 387). It then follows from (1.11) and (1.12) that

$$\lim_{t \rightarrow \infty} \text{Cov}(Y_{t-k}^2, Y_t^2) = \exp\left[2\mu_h + \sigma_h^2\right] \left[\exp\left(\sigma_h^2 \gamma_1^k\right) - 1\right],\quad (1.13)$$

see also Mills, 1999, p.128, Anderson and Sorenson 1996, p.331 and Jacquire et.al, 1994, p. 387. Consequently, by applying (1.10) and (1.13), one obtains the asymptotic lag k ($k = 1, \dots, t-1$) correlation between the squared responses Y_{t-k}^2 and Y_t^2 as

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Corr}[Y_{t-k}^2, Y_t^2] &= \lim_{t \rightarrow \infty} \left[\frac{\text{Cov}[Y_{t-k}^2, Y_t^2]}{\sqrt{\text{Var}(Y_{t-k}^2) \text{Var}(Y_t^2)}} \right] \\ &= \frac{\exp(2\mu_h + \sigma_h^2) \left(\exp[\gamma_1^k \sigma_h^2] - 1 \right)}{\exp[2\mu_h + \sigma_h^2] [3 \exp(\sigma_h^2) - 1]} = \frac{\left(\exp[\gamma_1^k \sigma_h^2] - 1 \right)}{[3 \exp(\sigma_h^2) - 1]}. \end{aligned} \quad (1.14)$$

Note that, the numerator in (1.14) lies between 0 to ∞ and the denominator lies between 2 to ∞ . Hence, the asymptotic lag k correlation between Y_{t-k}^2 and Y_t^2 is bounded between zero to 1. That is,

$$0 < \lim_{t \rightarrow \infty} \text{Corr}[Y_{t-k}^2, Y_t^2] < 1.$$

Asymptotic Kurtosis:

Note that it is standard to use the kurtosis to explain the volatility in the data. By definition, the limiting ($t \rightarrow \infty$) kurtosis under the SV model has the formula given by

$$\lim_{t \rightarrow \infty} \kappa(Y_t) = \lim_{t \rightarrow \infty} \frac{E[Y_t^4]}{(E[Y_t^2])^2} = \lim_{t \rightarrow \infty} \frac{3E[\sigma_t^4]}{(E[\sigma_t^2])^2} = 3 \exp(\sigma_h^2) > 3, \quad (1.15)$$

[see also Harvey 1994, p.249, Mills, 1999, p.128]. Hence, the volatility model (1.2) produces larger kurtosis when compared to the Gaussian kurtosis. Also, the peak

appears to depend on the volatility parameter values γ_1 and σ_η^2 (see also 1.15). Thus, it is essential to estimate the model (1.2) parameters, namely γ_1 and σ_η^2 consistently and efficiently.

1.1.2 ARCH/GARCH Models

As opposed to the above model, in 1982, Engle suggested an observation driven model, that is, ARCH model, to study the time varying observed variances. In the ARCH model, the conditional variance of the time series $\{y_t\}$ is a deterministic function of lagged values of the squared observations [Engle, 1982]. That is,

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \dots + \alpha_p y_{t-p}^2 \quad t = 1, 2, \dots, T. \quad (1.16)$$

Further, Bollerslev (1986) generalized the ARCH model, by expressing the conditional variance as a function of lagged squared observations and lagged variances. That is,

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \dots + \alpha_p y_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \dots + \beta_q \sigma_{t-q}^2 \quad t = 1, 2, \dots, T, \quad (1.17)$$

which is referred to as the generalized ARCH (GARCH) model.

However, in the thesis, we concentrate only on the SV model (1.1) - (1.2) and a generalization to be discussed in Chapter 4. Thus, there will be no further discussion of the ARCH/GARCH models.

We now turn back to the SV model and briefly discuss two widely used techniques for the estimation of the parameters (γ_1 and σ_η^2) of this SV model.

1.2 Two Existing Estimation Methods for SV Model

There exist many approaches for the estimation of the volatility parameters namely, γ_0, γ_1 and σ_η^2 , involved in the SV model (1.2). For example, we refer to the (1) generalized method of moments (GMM) [Melino and Turnbull, 1990, Anderson and Sorenson 1996], (2) Quasi maximum likelihood (QML) [Harvey et.al 1994, Ruiz, 1994], (3) simulation-based maximum likelihood (SML)[Danielsson 1994 and Danielsson and Richard 1993], and (4) Bayesian Markov chain Monte Carlo (MCMC) analysis approach [Jacquier et.al 1994]. Since the GMM and QML approaches are computationally less cumbersome as compared to the SML and MCMC approaches, they have been widely used over the last three decades, especially in a large time series set up. For recent discussion on these two methods, we for example, refer to Anderson and Sorenson (1997) and Ruiz (1997). For convenience, these two approaches are presented in brief in the following two sections.

1.2.1 Generalized Method of Moments (GMM) approach

The GMM approach (Hansen (1982)) utilizes a large number of unbiased moment functions of the absolute and/or squared responses. More specifically, Anderson and

Sorensen (1996, p. 350-351) have used 34 unbiased moment functions (see also Anderson and Sorensen (1997)) to construct the GMM estimating equations mainly for γ_1 and σ_η^2 parameters. These 34 moment functions are given as

$$\begin{aligned} g_{e1} &= |y_t| - E|y_t|, & g_{e2} &= y_t^2 - E[y_t^2], & g_{e3} &= |y_t|^3 - E|y_t|^3, \\ g_{e4} &= y_t^4 - E[y_t^4], & g_{t,4+l} &= |y_t y_{t-l}| - E[|y_t y_{t-l}|], & g_{t,14+l} &= y_t^2 y_{t-l}^2 - E[y_t^2 y_{t-l}^2], \\ g_{t,24+l} &= |y_t| y_{t-l}^2 - E[|y_t| y_{t-l}^2] & l &= 1, \dots, 10. \end{aligned} \quad (1.18)$$

and they are used to construct the GMM estimating equation for $\alpha^* = (\gamma_0, \gamma_1, \sigma_\eta^2)'$ as given by

$$\frac{\partial g'(\alpha^*)}{\partial \alpha^*} \Lambda^{-1} g(\alpha^*) = 0, \quad (1.19)$$

where

$$g(\alpha^*) = \frac{1}{T} \sum_{t=1}^T g_t(\alpha^*) \quad \text{with} \quad g_t(\alpha^*) = [g_{t1}(\alpha^*), \dots, g_{t34}(\alpha^*)]'. \quad (1.20)$$

and $\frac{\partial g'(\alpha^*)}{\partial \alpha^*}$ is the 3×34 first derivative matrix, and Λ is the 34×34 weight matrix with $\Lambda = \text{Cov}(g(\alpha^*))$ as an optimal choice. Later on, Anderson and Sorensen (1997, section 3, p.399-400) have used 24 unbiased moment functions out of 34 functions shown in (1.18). Their 24 functions are:

$$\begin{aligned} g_{e1} &= |y_t| - E|y_t|, & g_{e2} &= y_t^2 - E[y_t^2], & g_{e3} &= |y_t|^3 - E|y_t|^3, & g_{e4} &= y_t^4 - E[y_t^4] \\ g_{t,4+l} &= |y_t y_{t-l}| - E[|y_t y_{t-l}|], & g_{t,14+l} &= y_t^2 y_{t-l}^2 - E[y_t^2 y_{t-l}^2], & l &= 1, \dots, 10. \end{aligned} \quad (1.21)$$

It should be clear from (1.18) and (1.21) that there is no guidelines available how the moment functions such as 34 functions in (1.18) and 24 functions in (1.21) were chosen, when in fact, one can think of infinite number of such functions (Melino and Turnbull (1990, p. 250)). This raises a concern about such an estimation procedure where an arbitrary large number of functions are needed to estimate a small number of parameters.

Furthermore, since the construction of the moment functions $g_{ij}(j = 1, \dots, 24 \text{ or } 34)$ requires the computation of expectation of different functions which may not be easy to simplify, and because the computation of the weight matrix Λ can be complicated, the GMM approach on the whole becomes very cumbersome. We, therefore, do not include this approach for the comparison with our proposed approaches that we discuss in Chapter 2.

1.2.2 Existing Quasi Maximum Likelihood (QML) approach

As an alternative to the GMM approach, there also exist a QML (Quasi maximum likelihood) approach for the estimation of the parameters of the SV model (1.1) - (1.2). For example, we refer to [Nelson(1988), Ruiz (1994), Harvey et.al (1994), and Mills (1999, p.130-131)]. This QML method is developed first by formulating a quasi (pseudo) likelihood (QL) based on normal approximation to the log chi-square distribution of $u_t = \log \epsilon_t^2 - E[\log \epsilon_t^2]$, where $\epsilon_t \stackrel{iid}{\sim} N(0, 1)$, and then maximizing this

quasi-likelihood with respect to the desired parameters. Note that this QL abbreviation may confuse the QL used in the generalized linear model (GLM) set up, where QL is constructed by using the first two moments of the data. Due to this approximation the QML approach is supposed to lose efficiency [Broto and Ruiz (2004)] in estimating the parameters. This approach is, however, not so cumbersome as compared to the GMM approach.

We now present the QML approach in brief. For this purpose, we re-express the model (1.1) as

$$\begin{aligned} z_t = \log y_t^2 &= \log \sigma_t^2 + \log \epsilon_t^2 \\ &= E[\log \epsilon_t^2] + \log \sigma_t^2 + u_t \\ &\equiv \kappa_1 + \log \sigma_t^2 + u_t \quad t = 1, \dots, T, \end{aligned} \quad (1.22)$$

where $\epsilon_t \stackrel{iid}{\sim} N(0, 1)$ and $\kappa_1 = -1.27$. Further, u_t follows the log chi-square distribution with mean zero and variance $\kappa_2 = \pi^2/2$ [Abramovitz and Stegun (1970, p. 943)]. It then follows that the exact likelihood function for γ_1 and σ_η^2 is given by

$$\begin{aligned} L(\gamma_1, \sigma_\eta^2 | z_1, z_2, \dots, z_T) &= \int_{\sigma_1^2, \dots, \sigma_T^2} \prod_{t=1}^T g(z_t - \kappa_1 - \ln \sigma_t^2) \\ &\quad f(\sigma_1^2) \prod_{t=2}^T f(\sigma_t^2 | \sigma_{t-1}^2) d\sigma_1^2 \dots d\sigma_T^2 \end{aligned} \quad (1.23)$$

where $g(u_t)$ represents the $\log \chi^2(0, \kappa_2)$ distribution.

It is, however, clear from (1.23) that the integration over the random variances $\sigma_1^2, \dots, \sigma_T^2$ is complex mainly because they follow the dynamic relationship (1.2).

Some authors have used an alternative 'working' ML approach, namely, a quasi-maximum likelihood (QML) approach. See, for example, Nelson(1988), Harvey et.al (1994), Ruiz (1994), Koopman et al (1995, chapter 7.5) and Mills (1999, p 130-131). Specially, to approximate the exact likelihood function in (1.23), Ruiz (1994) and Harvey et.al (1994), for example, have approximated the distribution by pretending that $\mathbf{z} = (z_1, z_2, \dots, z_T)'$ follows a quasi-multivariate normal distribution. This leads to a quasi-likelihood, which is maximized to obtain the QML estimates for γ_1 and σ_0^2 . This approach is computationally feasible for the estimation of the required parameters, specially as compared to the GMM approach. For this reason, we will include the QML approach for asymptotic variance comparison with our proposed estimates. This comparison will be done in Chapter 3.

1.3 Objective of The Thesis

Even though the volatility models are very important to study the dynamic changes in variances in a time series, and also these models are widely used, there is no user friendly (simple) estimation techniques available for the inferences in stochastic volatility model. This is because, as explained in Sections 1.2.1 and 1.2.2, the existing GMM approach is arbitrary and cumbersome, whereas the QML approach may not be efficient (as compared to other simpler approaches) even if it is known to be

feasible computationally.

One of the main objectives of the thesis is to develop a simpler and efficient estimation approach, specially as compared to the QML approach, given that the GMM approach is very cumbersome and hence it is not practical. Furthermore, there are many situations where it may be appropriate to consider correlated observations conditional on the variances whereas in the SV model (1.1)-(1.2), the responses $\{y_t\}$ are uncorrelated conditional on $\{\sigma_t\}$. In non-volatility set up, this type of correlation models for observations (dynamic model) has been discussed by some authors. For example, we refer to Bun and Carree (2005), and Rao, Sutradhar and Pandit (2010) in the longitudinal set up. In the thesis, we consider this type of dynamic model for time series observations, as opposed to the longitudinal observations, conditional on the heteroscedastic errors of the series.

In Chapter 2, we propose two new simpler estimation approaches as compared to the existing approaches. These new approaches are developed by using only few appropriately selected unbiased moment functions, and they will be referred to as the simple method of moments (SMM) and ‘working’ generalized quaslikelihood (WGQL) approaches. It is argued that as opposed to the existing GMM approach using arbitrarily selected 24 or 34 unbiased moment functions, for example, it is enough to consider only 2 or 3 unbiased moments to construct the proposed SMM and WGQL

estimating equations. However, the important task is to find the best way to solve the estimating equations to be constructed by using these few moment functions. The construction of the SMM approach both for finite and asymptotic cases is discussed in details. However, for the WGQL approach, we provide the construction in details for the finite case only. The construction for the asymptotic case can be done easily. Numerical algorithms are also provided to make these approaches user friendly. To examine the asymptotic behavior of the proposed SMM and WGQL approaches, in this chapter, we provide an asymptotic efficiency comparison between these two approaches. Furthermore, since the existing QML approaches is computationally manageable, we have included this approach in our asymptotic efficiency comparison.

Based on the numerical algorithm developed in Chapter 2, in Chapter 3, we conduct a simulation study, first to examine the finite sample performances of the proposed SMM and WGQL approaches. Next, we continue the simulation study to examine their large sample performances. For this purpose, we provide both simulated mean and standard errors of the proposed estimators. Note that these large sample based simulated standard errors are comparable with the standard errors reported in Chapter 2. In same chapter, the effects of estimation of the volatility parameters on the kurtosis are examined for small as well as moderately large time series.

In Chapter 4, we extend the SV model in (1.1)-(1.2) to an observation driven

dynamic model set up. This extended model, unlike the SV model, can accommodate correlated responses conditional on the non-stationary variances of the series. For simplicity, we will however, consider the lag 1 conditional dependence among the observations conditional on the variances. This generalized model will be referred to as the observation-driven dynamic dynamic (ODDD) volatility model. The proposed SMM approach will be used to estimate the parameters of this ODDD volatility model, whereas the regression effects and dynamic dependence parameter will be estimated through a generalized quasi-likelihood (GQL) approach.

The thesis is concluded in Chapter 5, with some remarks on possible future works in related areas.

Chapter 2

Proposed Estimation Technique in Stochastic Volatility Models

Note that because of the importance of volatility model (1.1)-(1.2), there has been an enormous effort, in the past to obtain consistent and efficient estimates of the parameters of this model. As mentioned in the last chapter, we refer to the GMM, QML, SML and MCMC methods for the estimation of the parameters involved in the SV model. Note however that among all these approaches, the GMM and QML approaches are still widely followed in practice even though these approaches are either complex and arbitrary. Also, there is no guarantee that one method will be more efficient than the other (see for example, Anderson and Sorensen (1997, Sections 4-5), Ruiz (1997)). The relative performance of the GMM and QML approaches is given

mainly because of the fact that other approaches are either computationally more involved or less efficient than these approaches.

Since the GMM and QML approaches are still considered to be complicated, in this chapter we investigate for any possible simpler estimation approaches. More specifically, in Section 2.1 we develop a moment technique which, unlike the GMM approach, uses only two unbiased moment functions to construct the estimating equations for two important volatility parameters of the SV model. As mentioned earlier, we refer to this method as the SMM (Simple Method of Moments) approach. In Section 2.2, we provide a similar but different approach, namely, a 'working' generalized quaslikelihood (WGQL) approach. In Section 2.4, we compute the asymptotic variances of these SMM and WGQL estimators, which are, subsequently, used in Section 2.5 for a numerical comparison. Also the variances of the estimators are compared with the modified QML approach. Note that the GMM approach will not be considered for comparison, as it was indicated in (1.18)-(1.19) that it uses arbitrarily large number of moment functions, which is not user friendly.

2.1 A Simple Method of Moments (SMM)

For simplicity, similar to Ruiz (1994), Anderson and Sorenson (1997) and Broto and Ruiz (2004), we choose $\gamma_0 = 0$ under the volatility model (1.2). Similarly, even though $\ln \sigma_t^2$ under the SV model, (1.2) is supposed to be a random $N\left(0, \frac{\sigma_\eta^2}{1 - \gamma_1^2}\right)$

variable, for convenience, one may choose a small value for σ_1^2 such that $\ln \sigma_1^2 \rightarrow 0$. Now, for the construction of the moment estimating equations for the main parameters, namely γ_1 and σ_η^2 , we choose only two unbiased moment functions as shown in Section 2.1.1. A justification for selecting two such moment functions is also outlined.

2.1.1 Unbiased Moment Estimating Equations for γ_1 and σ_η^2 in Finite Time Series

2.1.1.1 Selection of Moment Function for Estimating σ_η^2

Note that it follows from the model (1.1) - (1.2) that if $h_t = \log \sigma_t^2$ were following a white noise series with mean 0 and variance σ_η^2 , that is $E[\sigma_t^2] = \text{Var}(Y_t) = h^*(\sigma_\eta^2)$, a suitable constant function of σ_η^2 , then one would have estimated $h^*(\sigma_\eta^2)$ consistently by using $S_1 = \frac{1}{T} \sum_{t=1}^T [y_t - E(Y_t)]^2$. This is because

$$E[S_1] = \frac{1}{T} \sum_{t=1}^T \text{Var}(Y_t) = h^*(\sigma_\eta^2). \quad (2.1)$$

Note that as $E[Y_t] = 0$ [see also 1.4], S_1 has the simple form as $S_1 = \frac{1}{T} \sum_{t=1}^T y_t^2$. However, under the present model, σ_t^2 's are unobservable and their log values satisfy a non stationary Gaussian AR(1) type relationship given by (1.2), with errors $\eta_t \stackrel{\text{ind}}{\sim} N(0, \sigma_\eta^2)$. This leads to the expectation of S_1 as a function of both γ_1 and σ_η^2 , instead of $h^*(\sigma_\eta^2)$.

Suppose that

$$E[S_1] = g_1(\gamma_1, \sigma_\eta^2, \sigma_1^2), \quad (2.2)$$

for a suitable known function 'g₁'. We evaluate this g₁(.) function in Theorem 2.1.1 below.

Note that between γ_1 and σ_η^2 involved in g₁(.), γ_1 is known to be a bounded parameter. That is, $|\gamma_1| < 1$. This assumption makes the AR(1) process for $\ln \sigma_t^2 = h_t$ to be stationary. But, unlike γ_1 , $\sigma_\eta^2 > 0$ can take any value in the real line $[0, \infty]$.

However, since $E[S_1] = h^*(\sigma_\eta^2)$ in the white noise case, we suggest to exploit S_1 for the construction of the estimating equation for σ_η^2 , even if the series is not white noise. This is because, the desired estimating equation should also be valid for the white noise case. Thus $S_1 - g_1(\cdot)$ would be considered as the best possible unbiased moment function for the estimation of σ_η^2 . That is, we solve the SMM estimating equation

$$S_1 - g_1(\gamma_1, \sigma_\eta^2, \sigma_1^2) = 0, \quad (2.3)$$

for σ_η^2 , by using a suitable value for γ_1 such that $|\gamma_1| < 1$. We now return to the derivation of g₁(.) as in the following theorem, before we provide the selection for the unbiased moment function for the other parameter γ_1 .

Theorem 2.1.1. *The unconditional expectation for S₁ is given by*

$$E[S_1] = g_1(\gamma_1, \sigma_\eta^2, \sigma_1^2),$$

where

$$g_1(\gamma_1, \sigma_\eta^2, \sigma_1^2) = \frac{1}{T} \left[\sigma_1^2 + \sum_{t=2}^T \exp \left(\gamma_1^{t-1} \ln \sigma_1^2 + \frac{\sigma_\eta^2}{2} \sum_{r=0}^{t-2} \gamma_1^{2r} \right) \right]. \quad (2.4)$$

Proof. Since $S_1 = \frac{1}{T} \sum_{t=1}^T y_t^2$, we write its unconditional expectation as

$$\begin{aligned} E[S_1] &= E_{\sigma_{(r)}^2} E \left[\frac{1}{T} \sum_{t=1}^T Y_t^2 | \sigma_t^2 \right] = \frac{1}{T} E_{\sigma_{(r)}^2} E[Y_1^2] + \sum_{t=2}^T E_{\sigma_{(r)}^2} E[Y_t^2 | \sigma_t^2] \\ &= \frac{1}{T} \left[E_{\sigma_{(r)}^2} E[Y_1^2 | \sigma_1^2] + E_{\sigma_{(r)}^2} E \left[\sum_{t=2}^T Y_t^2 | \sigma_t^2 \right] \right] \\ &= \frac{1}{T} \left[E_{\sigma_{(r)}^2} E[\epsilon_1^2 \sigma_1^2 | \sigma_1^2] + \sum_{t=2}^T E_{\sigma_{(r)}^2} E[\epsilon_t^2 \sigma_t^2 | \sigma_t^2] \right] \\ &= \frac{1}{T} \left[E_M[\sigma_1^2] + \sum_{t=2}^T E_M[\sigma_t^2] \right], \end{aligned} \quad (2.5)$$

with $\sigma_{(T)}^2 = (\sigma_1^2, \dots, \sigma_T^2)$, where E_M denotes the model based expectation, where the assumed model for σ_t^2 is given by (1.2). Now for a known value of $\sigma_1^2 = \sigma_{j_0}^2$ such that $\ln \sigma_1^2 \rightarrow 0$, we rewrite the expectation in (2.5) as

$$E[S_1] = \frac{\sigma_{j_0}^2}{T} + \frac{1}{T} \sum_{t=2}^T E_M[\sigma_t^2]. \quad (2.6)$$

We now derive $E_M[\sigma_t^2]$ for (2.6). First by using the recurrence relationship from the equation (1.2), we write the general form for σ_t^2 as

$$\begin{aligned} \sigma_t^2 &= \exp \left(\gamma_1^{t-1} \ln \sigma_{j_0}^2 + \gamma_1^{t-2} \eta_2 + \gamma_1^{t-3} \eta_3 + \gamma_1^{t-4} \eta_4 + \dots + \gamma_1 \eta_{t-1} + \eta_t \right) \\ &= \exp \left(\gamma_1^{t-1} \ln \sigma_{j_0}^2 + \sum_{r=0}^{t-2} \gamma_1^r \eta_{(t-r)} \right) \quad t = 2, \dots, T. \end{aligned} \quad (2.7)$$

Next, since $\eta_t \stackrel{iid}{\sim} N(0, \sigma_\eta^2)$, by using normal moment generating function $E(e^\eta) =$

$\exp(\sigma_0^2/2)$, it follows from (2.7) that $E_M[\sigma_t^2]$ is given by

$$\begin{aligned} E_M[\sigma_t^2] &= \exp\left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \frac{1}{2} \sigma_0^2 \sum_{r=0}^{t-2} \gamma_1^{2r}\right) \\ &= \exp\left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \frac{\sigma_0^2}{2} \left\{ \frac{1 - \gamma_1^{t-1}}{1 - \gamma_1} \right\}\right) \quad t = 2, \dots, T. \end{aligned} \quad (2.8)$$

Now by using $E_M[\sigma_t^2]$ from (2.8) in (2.5), we obtain $E[S_1] = g_1(\cdot)$ as in the theorem. \square

Note that even if γ_1 is known, the solution of (2.3) requires a good initial value for σ_0^2 , which we suggest to obtain by solving an asymptotic unbiased estimating equation whereas the estimating equation in (2.3) is valid for any $t \geq 2$. Note that for $t \rightarrow \infty$, $|\gamma_1|^{t-1} \rightarrow 0$ as $|\gamma_1| < 1$. It then follows that

$$\begin{aligned} \lim_{t \rightarrow \infty} E[Y_t^2] &= \lim_{t \rightarrow \infty} E_M[\sigma_t^2] \\ &= \exp\left[\frac{\sigma_0^2}{2} \left(\frac{1}{1 - \gamma_1^2}\right)\right], \\ &= g_{10}(\gamma_1, \sigma_0^2). \end{aligned} \quad (2.9)$$

We now want to construct an unbiased moment function as a reflection of the limiting property shown in (2.9). For this, one can find a T_0 such that for any $t > T_0$, $\gamma_1^{t-1} \rightarrow 0$ for $|\gamma_1| < 1$ and write a basic statistic as

$$S_{10} = \frac{1}{T - T_0} \sum_{t=T_0+1}^T y_t^2 \quad (2.10)$$

as a modification to the formula for $S_1 = \frac{1}{T} \sum_{t=1}^T y_t^2$ used in the theorem.

It then follows that

$$\begin{aligned}\lim_{T \rightarrow \infty} E[S_{10}] &= \frac{1}{T - T_0} \sum_{t=T_0+1}^T \lim_{t \rightarrow \infty} E[Y_t^2] \\ &= \exp\left[\frac{\sigma_\eta^2}{2} \left(\frac{1}{1 - \gamma_1^2}\right)\right] = g_{10}(\gamma_1, \sigma_\eta^2).\end{aligned}\quad (2.11)$$

This asymptotic expectation is quite simple for the derivation of an initial value for σ_η^2 , for an initial value of $\gamma_1 = \gamma_1(0)$. For $\gamma_1 = \gamma_1(0)$, let $\sigma_\eta^2(0)$ be the solution of

$$S_{10} - g_{10}(\gamma_1, \sigma_\eta^2) = 0. \quad (2.12)$$

That is

$$\sigma_\eta^2(0) = 2 \ln S_{10}(1 - \gamma_1^2(0)). \quad (2.13)$$

2.1.1.2 Selection of Moment Function for Estimating γ_1

Next, to construct an unbiased estimating equation for γ_1 , we first observe that γ_1 is the lag 1 dependence parameter in the Gaussian AR(1) model (1.2). We therefore, choose a lag 1 based function given by $S_2 = \frac{1}{T-1} \sum_{t=2}^T y_{t-1}^2 y_t^2$ to construct the moment equation for γ_1 . Suppose that the expectation of S_2 as a function of both γ_1 and σ_η^2 is denoted by $g_2(\gamma_1, \sigma_\eta^2, \sigma_{10}^2)$. One may then solve the SMM estimating equation for γ_1 given by

$$S_2 - g_2(\gamma_1, \sigma_\eta^2, \sigma_{10}^2) = 0, \quad (2.14)$$

for known value of $\sigma_\eta^2 = \sigma_\eta^2(0)$. We now derive the formula for $g_2(\gamma_1, \sigma_\eta^2, \sigma_{10}^2)$ which is given in Theorem 2.1.2 below.

Theorem 2.1.2. *The unconditional expected value of S_2 is*

$$E[S_2] = g_2(\gamma_1, \sigma_\eta^2, \sigma_{10}^2)$$

where

$$\begin{aligned} g_2(\gamma_1, \sigma_\eta^2, \sigma_1^2) &= \frac{1}{T-1} \left[\sigma_{10}^2 \exp\left(\gamma_1 \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2}\right) \right. \\ &\quad \left. + \sum_{t=3}^T \exp\left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \gamma_1^{t-2} \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2} \{(1+\gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1\}\right) \right]. \end{aligned} \quad (2.15)$$

Proof. Since $S_2 = \frac{1}{T-1} \sum_{t=2}^T y_{t-1}^2 y_t^2$, we write its unconditional expectation as

$$\begin{aligned} E[S_2] &= E_{\sigma_{(T)}} E \left[\frac{1}{T-1} \sum_{t=2}^T y_{t-1}^2 y_t^2 | \sigma_{t-1}^2 \sigma_t^2 \right] = \frac{1}{T-1} E_{\sigma_{(T)}} E \left[y_1^2 y_2^2 + \sum_{t=3}^T y_{t-1}^2 y_t^2 | \sigma_{t-1}^2 \sigma_t^2 \right] \\ &= \frac{1}{T-1} \left[E_{\sigma_{(T)}} E[y_1^2 y_2^2 | \sigma_1^2 \sigma_2^2] + E_{\sigma_{(T)}} E \left[\sum_{t=3}^T y_{t-1}^2 y_t^2 | \sigma_{t-1}^2 \sigma_t^2 \right] \right] \\ &= \frac{1}{T-1} \left[E_{\sigma_{(T)}} E[\epsilon_1^2 \epsilon_2^2 \sigma_1^2 \sigma_2^2 | \sigma_1^2 \sigma_2^2] + \sum_{t=3}^T E_{\sigma_{(T)}} E[\epsilon_{t-1}^2 \epsilon_t^2 \sigma_{t-1}^2 \sigma_t^2 | \sigma_{t-1}^2 \sigma_t^2] \right] \\ &= \frac{1}{T-1} \left[E[\sigma_1^2 \sigma_2^2] + \sum_{t=3}^T E[\sigma_{t-1}^2 \sigma_t^2] \right] = \frac{1}{T-1} \left[\sigma_{10}^2 E_M[\sigma_2^2] + \sum_{t=3}^T E_M[\sigma_{t-1}^2 \sigma_t^2] \right], \\ &= \frac{1}{T-1} \left[\sigma_{10}^2 \left[\exp\left(\gamma_1 \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2}\right) \right] \right. \\ &\quad \left. + \sum_{t=3}^T \exp\left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \gamma_1^{t-2} \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2} \{(1+\gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1\}\right) \right] \\ &= \frac{1}{T-1} \left[\sigma_{10}^2 \left[\exp\left(\gamma_1 \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2}\right) \right] + g_p(\gamma_1, \sigma_\eta^2) \right] \quad (\text{say}), \end{aligned} \quad (2.16)$$

with $\sigma_{(T)}^2 = (\sigma_1^2, \dots, \sigma_T^2)$.

Note that the derivation for $E_M[\sigma_{t-1}^2 \sigma_t^2]$, the expectation of the pairwise products of

σ_{t-1}^2 and σ_t^2 is lengthy but straightforward. For convenience, we provide the derivation for $E_M[\sigma_{t-1}^2 \sigma_t^2]$ in Appendix A. For convenience, we, re-write the formula for $E_M[\sigma_{t-1}^2 \sigma_t^2]$ from the appendix. The formula is:

$$E_M[\sigma_{t-1}^2 \sigma_t^2] = \exp\left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \gamma_1^{t-2} \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2} \left\{ (1 + \gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1 \right\}\right). \quad (2.17)$$

□

Note that, in the above discussion, we have given a justification for the selection of two unbiased moment functions in (2.3) and (2.14) for the estimation of σ_η^2 and γ_1 respectively. We have also indicated that a good initial value of $\sigma_\eta^2 = \sigma_\eta^2(0)$ can be computed from the asymptotic moment equation for σ_η^2 given in (2.13). To make this estimation approach user friendly, we now give a numerical algorithm for solving (2.3) and (2.14) for σ_η^2 and γ_1 respectively, by using the initial value of $\sigma_\eta^2(0)$ obtained from (2.13).

Algorithm

Step 1: For a small initial value $\gamma_1 = \gamma_1(0)$ and $\sigma_1^2 = \sigma_{10}^2$ we choose $\sigma_\eta^2(0)$, an initial value of σ_η^2 by (2.13).

Step 2: Once the initial values are chosen/computed as in Step 1, we solve $S_2 - g_2(\gamma_1, \sigma_\eta^2, \sigma_{10}^2) = 0$ by (2.14) iteratively to obtain an improved value for γ_1 . The

iterative equation has the form

$$\hat{\gamma}_1(r+1) = \hat{\gamma}_1(r) + \left[\left(\frac{\partial g_2(\gamma_1, \sigma_q^2, \sigma_{10}^2)}{\partial \gamma_1} \right)^{-1} \left(S_2 - g_2(\gamma_1, \sigma_q^2, \sigma_{10}^2) \right) \right]_{|\gamma_1(r)} \quad (2.18)$$

where $\hat{\gamma}_1(r)$ is a value of γ_1 at r^{th} iteration, and $[\cdot]_{|\gamma_1(r)}$ is the value of the expression in the squared bracket evaluated at $\gamma_1 = \hat{\gamma}_1(r)$. By following the formula for $g_2(\gamma_1, \sigma_q^2, \sigma_{10}^2)$ from (2.15), the derivative of $\frac{\partial g_2(\gamma_1, \sigma_q^2, \sigma_{10}^2)}{\partial \gamma_1}$ in (2.18) has the formula given by

$$\begin{aligned} \frac{\partial \{g_2(\gamma_1, \sigma_q^2, \sigma_{10}^2)\}}{\partial \gamma_1} &= \frac{1}{T-1} \left[\sigma_{10}^2 \exp \left(\gamma_1 \ln \sigma_{10}^2 + \frac{\sigma_q^2}{2} \right) \gamma_1 \right. \\ &\quad + \left(\sum_{t=3}^T \exp \left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \gamma_1^{t-2} \ln \sigma_{10}^2 + \frac{\sigma_q^2}{2} \{ (1 + \gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1 \} \right) \right. \\ &\quad \left. \left((t-1) \gamma_1^{t-2} \ln \sigma_{10}^2 + (t-2) \gamma_1^{t-3} \ln \sigma_{10}^2 \right. \right. \\ &\quad \left. \left. + \frac{\sigma_q^2}{2} \{ 2(1 + \gamma_1) \sum_{l=0}^{t-3} \gamma_1^{2l} + (1 + \gamma_1)^2 \sum_{l=0}^{t-3} (2l) \gamma_1^{(2l-1)} \} \right) \right] \quad (2.19) \end{aligned}$$

Step 3: The estimate of γ_1 obtained from Step 2 is then used to solve $S_1 - g_1(\gamma_1, \sigma_q^2, \sigma_{10}^2) = 0$ in (2.3) iteratively to obtain an improvement over $\sigma_q^2(0)$. The iterative equation has the form

$$\hat{\sigma}_q^2(r+1) = \hat{\sigma}_q^2(r) + \left[\left(\frac{\partial g_1(\gamma_1, \sigma_q^2, \sigma_{10}^2)}{\partial \sigma_q^2} \right)^{-1} \left(S_1 - g_1(\gamma_1, \sigma_q^2, \sigma_{10}^2) \right) \right]_{|\sigma_q^2(r)} \quad (2.20)$$

where $\hat{\sigma}_q^2(r)$ is the value of σ_q^2 at r^{th} iteration, and $[\cdot]_{|\sigma_q^2(r)}$ is the value of the expression in the square bracket evaluated at $\sigma_q^2 = \hat{\sigma}_q^2(r)$. By following the formula for $g_1(\gamma_1, \sigma_q^2, \sigma_{10}^2)$ from (2.4), the derivative of $\frac{\partial g_1(\gamma_1, \sigma_q^2, \sigma_{10}^2)}{\partial \sigma_q^2}$ in (2.20) has the formula

given by

$$\frac{\partial g_1(\gamma_1, \sigma_\eta^2, \sigma_1^2)}{\partial \sigma_\eta^2} = \frac{1}{T} \left[\sum_{t=2}^T \exp\left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2} \sum_{r=0}^{t-2} \gamma_1^{2r}\right) \left(\frac{1}{2} \sum_{r=0}^{t-2} \gamma_1^{2r}\right) \right]. \quad (2.21)$$

This 3 steps cycle of iteration continues until convergence. Let the final estimates obtained from (2.18) and (2.20) be denoted by $\hat{\gamma}_{1, SMM}$ and $\hat{\sigma}_{\eta, SMM}^2$ respectively.

2.1.2 Moment Estimation in Large Time Series

In this case we provide the estimating formulas for γ_1 and σ_η^2 by using $\{y_t\}$ for $t > T_0$, where T_0 is sufficiently large and $|\gamma_1| < 1$ leading $\gamma_1^{t-1} \rightarrow 0$. For this purpose, $\sigma_\eta^2 = \sigma_\eta^2(0)$ is still evaluated from (2.13) by solving $S_{10} - g_{10}(\cdot) = 0$ (2.12) where
$$S_{10} = \frac{1}{T - T_0} \sum_{t=T_0+1}^T y_t^2.$$

By the same token, we now consider $S_{20} = \frac{1}{T - T_0 - 1} \sum_{t=T_0+2}^T y_{t-1}^2 y_t^2$ and solve the estimating equation

$$S_{20} - g_{20}(\gamma_1, \sigma_\eta^2) = 0. \quad (2.22)$$

for γ_1 , where

$$\begin{aligned} g_{20}(\gamma_1, \sigma_\eta^2) &= \lim_{T \rightarrow \infty} E[S_{20}] = \frac{1}{T - T_0 - 1} \sum_{t=T_0+2}^T \lim_{t \rightarrow \infty} E[Y_{t-1}^2 Y_t^2] \\ &= \frac{1}{T - T_0 - 1} \sum_{t=T_0+2}^T \lim_{t \rightarrow \infty} E_M[\sigma_{t-1}^2 \sigma_t^2] \\ &= \exp\left[\frac{\sigma_\eta^2}{1 - \gamma_1}\right], \end{aligned} \quad (2.23)$$

where the formula of $E_M[\sigma_{t-1}^2 \sigma_t^2]$ is given in (2.17).

Algorithm for $T \rightarrow \infty$

As far as the algorithm for this large T_0 case is concerned, we summarize it as follows.

Step 1: For a small initial value $\gamma_1 = \gamma_1(0)$ and $\sigma_1^2 = \sigma_{10}^2$ we choose $\sigma_n^2(0)$, as an initial value of σ_n^2 by (2.13).

Step 2: Once the initial values are choosen/computed as in Step 1, we solve (2.22) for γ_1 iteratively to obtain an improved value for γ_1 . The iterative equation has the form

$$\begin{aligned}\hat{\gamma}_1(r+1) &= \hat{\gamma}_1(r) + \left[\frac{\partial g_{20}}{\partial \gamma_1} \right]^{-1} [S_{20} - g_{20}] \\ &= \hat{\gamma}_1(r) + \left[\exp\left(\frac{\sigma_n^2}{1 - \gamma_1}\right) \left(\frac{\sigma_n^2}{(1 - \gamma_1)^2}\right) \right]^{-1} [S_{20} - g_{20}]_{|\gamma_1}. \quad (2.24)\end{aligned}$$

where $\hat{\gamma}_1(r)$ is a value of γ_1 at r^{th} iteration, and $[.]_{|\gamma_1(r)}$ is the value of the expression in the square bracket evaluated at $\gamma_1 = \hat{\gamma}_1(r)$.

Step 3: We use improved γ_1 from (2.24) in Step 2 and solve (2.13) to obtain an improved asymptotic estimate for σ_n^2 .

This cycles of iteration continues until convergence.

2.2 A Generalized Quasi Likelihood (GQL) Method in Finite Time Series

In Section 2.1, we proposed a user friendly simple method of moment (SMM) approach to estimate the volatility parameters for both finite and large time series cases. However, there exists a relatively new, namely, the generalized quaslikelihood (GQL) approach [Sutradhar (2004), Mallick and Sutradhar (2008)] that yields efficient estimates. In Section 2.2.1 and 2.2.2 we show how to construct this GQL approach for σ_{η}^2 and γ_1 respectively. Note that even though we provide the theoretical formulas for the covariance matrix involved in the GQL estimating equations, it can however be very time consuming to compute the inverse of such covariance matrices, needed for solving the GQL estimating equations. To avoid this numerical complexity, we provide some approximations to the construction of covariance matrices involved in the equations. This will naturally yield approximate GQL estimates for the parameters. For convenience we refer to this approximate GQL approach as a 'working' GQL (WGQL) approach, and provide the estimating equations for σ_{η}^2 and γ_1 in Section 2.3.1 and in Section 2.3.2, respectively.

2.2.1 GQL Estimating Equation for σ_η^2

Note that in Section 2.1, $\sum_{t=1}^T y_t^2$ was equated to its expectation to construct an unbiased moment estimating equation in the SMM approach. In the GQL approach, the same squared responses are used but in a different way. To be specific, in this approach, a quadratic form in the distances of the squared responses and their expectation, is minimized with respect to the desired parameters. Let

$$\mathbf{u} = [y_1^2, \dots, y_T^2, \dots, y_T^2]'. \quad (2.25)$$

with its unconditional expectation as

$$\lambda = E[\mathbf{U}] = [\lambda_1, \dots, \lambda_t, \dots, \lambda_T]'. \quad (2.26)$$

Further, let Σ be the covariance matrix of \mathbf{u} . In this GQL approach, the quadratic distance function, namely

$$Q = (\mathbf{u} - \lambda)' \Sigma^{-1} (\mathbf{u} - \lambda) \quad (2.27)$$

is minimized with respect to σ_η^2 , to obtain the estimating equation for this parameter.

To be specific, the GQL estimating equation for σ_η^2 is given by

$$\frac{\partial \lambda'}{\partial \sigma_\eta^2} \Sigma^{-1} (\mathbf{u} - \lambda) = 0, \quad (2.28)$$

[Sutradhar (2004), Mallick and Sutradhar (2008)] where $\frac{\partial \lambda}{\partial \sigma_\eta^2}$ is the derivative of λ with respect to (w.r.t) σ_η^2 .

For (2.28), we now provide the formulas for λ and Σ as in the following two theorems.

Theorem 2.2.1. For known $\sigma_1^2 = \sigma_{10}^2$, the elements of the unconditional expectation λ are given by

$$\lambda_t = E[Y_t^2] = \begin{cases} \sigma_{10}^2 & \text{for } t=1 \\ \exp\left[\gamma_1^{t-1} \ln \sigma_{10}^2 + \frac{\sigma_1^2}{2} \sum_{r=0}^{t-2} \gamma_1^{2r}\right] & \text{for } t = 2, \dots, T. \end{cases} \quad (2.29)$$

Proof. Note that

$$\lambda_1 = E[Y_1^2] = E_M[\sigma_1^2] = \sigma_{10}^2, \quad (2.30)$$

by assumption that σ_1^2 is known.

Now, for $t=2, \dots, T$, we write

$$\lambda_t = E[Y_t^2] = E[\sigma_t^2 \epsilon_t^2].$$

Since σ_t^2 and ϵ_t^2 are independent and $\epsilon_t \stackrel{\text{iid}}{\sim} N(0, 1)$, one obtains

$$\lambda_t = E[\sigma_t^2] E[\epsilon_t^2] = E_M[\sigma_t^2]. \quad (2.31)$$

The formula for $E_M[\sigma_t^2]$ for $t=2, \dots, T$ is given by (2.8). Hence the theorem. \square

Theorem 2.2.2. Let the diagonal elements of Σ be $\sigma_{tt} = \text{Var}(Y_t^2)$ and the lag k off diagonal elements be $\sigma_{t-k,t} = \text{Cov}(Y_{t-k}^2, Y_t^2)$. The formulas for the variances are given by

$$\sigma_{t,t} = \text{Var}[Y_t^2] = \begin{cases} 3\sigma_{10}^4 - \lambda_1^2 & \text{for } t=1 \\ 3\exp\left[2\gamma_1^{t-1} \ln \sigma_{10}^2 + 2\sigma_1^2 \sum_{r=0}^{t-2} \gamma_1^{2r}\right] - \lambda_t^2 & \text{for } t = 2, \dots, T, \end{cases} \quad (2.32)$$

and for $k=1, \dots, t-1$ and $t=2, \dots, T$, the lag k covariances have the formulas as

$$\sigma_{t-k,t} = \text{Cov}[Y_{t-k}^2, Y_t^2] = \begin{cases} \sigma_{10}^2 \lambda_t - \lambda_1 \lambda_t & \text{for } t-k=1, \\ \exp \left[\gamma_1^{t-1} \ln \sigma_{10}^2 + \gamma_1^{t-k-1} \ln \sigma_{10}^2 \right. \\ \left. + \frac{\sigma_{10}^2}{2} \left((1+\gamma_1^k)^2 \sum_{l=0}^{t-k-2} \gamma_1^{2l} + \sum_{r=0}^{k-1} \gamma_1^{2r} \right) \right] - \lambda_{t-k} \lambda_t & \text{for } t=2, \dots, T. \end{cases} \quad (2.33)$$

where λ_t is given by (2.29).

Proof. To obtain σ_{tt} , we write

$$\begin{aligned} \sigma_{tt} = \text{Var}[Y_t^2] &= E[Y_t^4] - \left(E[Y_t^2] \right)^2 = E[\epsilon_t^4 \sigma_t^4] - \lambda_t^2 \\ &= 3 E_M[\sigma_t^4] - \lambda_t^2 \quad \text{for } t=1, \dots, T. \end{aligned} \quad (2.34)$$

This is because $E[\epsilon_t^4] = 3$. Note that for $t=1$, $\sigma_{11} = \text{Var}[Y_1^2] = 3\sigma_{10}^4 - \lambda_1^2$. For $t=2, \dots, T$, recall from (2.7), that

$$\sigma_t^2 = \exp \left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \sum_{r=0}^{t-2} \gamma_1^r \eta_{t-r} \right).$$

By some algebra, we can write

$$\sigma_t^4 = \exp \left(2 \gamma_1^{t-1} \ln \sigma_{10}^2 + 2 \sum_{r=0}^{t-2} \gamma_1^r \eta_{t-r} \right), \quad (2.35)$$

with $\sigma_1^2 = \sigma_{10}^2$. Next, since $\eta_h \stackrel{iid}{\sim} N(0, \sigma_\eta^2)$, by using normal moment generating function $E(e^{u\eta}) = \exp(\sigma_\eta^2 u^2/2)$, it follows from (2.35) that $E_M[\sigma_t^4]$ is given by

$$E_M[\sigma_t^4] = \exp \left(2 \gamma_1^{t-1} \ln \sigma_{10}^2 + 2 \sigma_\eta^2 \sum_{r=0}^{t-2} \gamma_1^{2r} \right) \quad t=2, \dots, T. \quad (2.36)$$

Now by using $E_M[\sigma_t^2]$ from (2.36) in (2.34), we obtain the $\text{Var}(Y_t^2) = \sigma_H$ as in theorem.

Next, the derive the lag $k=1, \dots, t-1$ and $t=2, \dots, T$, unconditional covariance between Y_{t-k}^2 and Y_t^2 . We write

$$\begin{aligned} \text{Cov}(Y_{t-k}^2, Y_t^2) &= E[Y_{t-k}^2 Y_t^2] - E[Y_{t-k}^2] E[Y_t^2] \\ &= E[\sigma_{t-k}^2 \sigma_t^2 \epsilon_{t-k}^2 \epsilon_t^2] - \lambda_{t-k} \lambda_t \\ &= E_M[\sigma_{t-k}^2 \sigma_t^2] E[\epsilon_{t-k}^2 \epsilon_t^2] - \lambda_{t-k} \lambda_t \\ &= E_M[\sigma_{t-k}^2 \sigma_t^2] - \lambda_{t-k} \lambda_t \end{aligned} \quad (2.37)$$

with $\epsilon_t \stackrel{\text{iid}}{\sim} N(0,1)$ and ϵ_t^2 and σ_t^2 are independent. Note that the derivation for $E[\sigma_{t-k}^2 \sigma_t^2]$ is lengthy, which is given in appendix A. For convenience, here we re-write the formula for $E_M[\sigma_{t-k}^2 \sigma_t^2]$ from the appendix. The formula is:

$$\begin{aligned} E_M[\sigma_{t-k}^2 \sigma_t^2] &= \exp\left(\gamma_1^{(t-k)-1} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2\right. \\ &\quad \left. + \frac{\sigma_\eta^2}{2} \left[(1 + \gamma_1^k)^2 \sum_{l=0}^{t-k-2} \gamma_1^{2l} + \sum_{r=0}^{k-1} \gamma_1^{2r} \right] \right), \end{aligned} \quad (2.38)$$

with $E[\sigma_1^2 \sigma_2^2] = \sigma_{10}^2 \lambda_2$, where λ_2 is computed from (2.29). Now by using this formula from (2.38) into (2.37), we obtain the lag k covariances between Y_{t-k}^2 and Y_t^2 as in theorem. \square

Computational Formula for σ_η^2 Estimate:

Note that λ and Σ in (2.28), are functions of γ_1 and σ_η^2 . Now, it follows from

(2.28) that for known γ_1 , the iterative equation for σ_η^2 may be expressed as

$$\hat{\sigma}_\eta^2(r+1) = \hat{\sigma}_\eta^2(r) + \left[\left(\frac{\partial \lambda'}{\partial \sigma_\eta^2} \Sigma^{-1} \frac{\partial \lambda}{\partial \sigma_\eta^2} \right)^{-1} \left(\frac{\partial \lambda'}{\partial \sigma_\eta^2} \Sigma^{-1} (u - \lambda) \right) \right]_{[r]}, \quad (2.39)$$

where $\frac{\partial \lambda}{\partial \sigma_\eta^2}$ is the derivative of λ w.r.t σ_η^2 . By ((2.29), the derivative has the formula

$$\frac{\partial \lambda_t}{\partial \sigma_\eta^2} = \exp \left[\gamma_1^{t-1} \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2} \sum_{r=0}^{t-2} \gamma_1^{2r} \right] \left[\frac{1}{2} \sum_{r=0}^{t-2} \gamma_1^{2r} \right], \quad (2.40)$$

for $t = 2, \dots, T$. For $t=1$ case, $\frac{\partial \lambda_1}{\partial \sigma_\eta^2} = 0$. In (2.39) $\hat{\sigma}_\eta^2(r)$ is the value of σ_η^2 at r^{th} iteration, and $[\cdot]_{\hat{\sigma}_\eta^2(r)}$ is the value of the expression in the square bracket evaluated at $\sigma_\eta^2 = \hat{\sigma}_\eta^2(r)$. Let the final estimate from (2.39) is denoted by $\hat{\sigma}_{\eta, GQL}^2$.

2.2.2 GQL estimating equation for γ_1

Note that in Section 2.2, $\sum_{t=2}^T y_{t-1}^2 y_t^2$ was equated to its expectation to construct an unbiased moment estimating equation for γ_1 . In the GQL approach, we use the same lag 1 pairwise squared responses but in a different way. To be specific, in this approach, a quadratic form in the distances of the lag 1 pairwise squared responses and their expectation, is minimized with respect to the γ_1 . Let

$$\mathbf{v} = [y_1^2 y_2^2, \dots, y_{t-1}^2 y_t^2, \dots, y_{T-1}^2 y_T^2]'. \quad (2.41)$$

and its unconditional expectation is given by

$$\psi = E[\mathbf{V}] = [\psi_{12}, \dots, \psi_{t-1,t}, \dots, \psi_{T-1,T}']', \quad (2.42)$$

where $\psi_{t-1,t} = E[y_{t-1}^2 y_t^2]$. Further, let Ω be the covariance matrix of \mathbf{v} . In the GQL approach, the quadratic distance function, namely

$$Q^* = (\mathbf{v} - \psi)' \Omega^{-1} (\mathbf{v} - \psi) \quad (2.43)$$

is minimized with respect to γ_1 , to obtain the GQL estimating equation for this parameter. To be specific, the GQL estimating equation for γ_1 has the form

$$\frac{\partial \psi'}{\partial \gamma_1} \Omega^{-1} (\mathbf{v} - \psi) = 0, \quad (2.44)$$

[Sutradhar (2004), Mallick and Sutradhar (2008)] where $\frac{\partial \psi'}{\partial \gamma_1}$ is the derivative w.r.t γ_1 .

We now provide the formulas for ψ and Ω in (2.44) in the following theorems.

Theorem 2.2.3. For known $\sigma_1^2 = \sigma_{10}^2$, the elements of the unconditional expectation are given by

$$\psi_{t-1,t} = E[Y_{t-1}^2 Y_t^2] = \begin{cases} \sigma_{10}^2 \lambda_2 & \text{for } t=2 \\ \exp \left[\gamma_1^{t-2} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2 \right. \\ \left. + \frac{\sigma_2^2}{2} \left((1 + \gamma_1)^2 \sum_{i=0}^{t-3} \gamma_1^{2i} + 1 \right) \right] & \text{for } t = 3, \dots, T. \end{cases} \quad (2.45)$$

Proof. Note that

$$\psi_{12} = E[Y_1^2 Y_2^2] = E[\sigma_1^2 \sigma_2^2] = \sigma_{10}^2 \lambda_2. \quad (2.46)$$

whereas for $t=3, \dots, T$, the expectations of the products of lag 1 squared observations are given by

$$\begin{aligned}\psi_{t-1,t} &= E[Y_{t-1}^2 Y_t^2] = E[\sigma_{t-1}^2 \epsilon_{t-1}^2 \epsilon_t^2] \\ &= E[\sigma_{t-1}^2, \sigma_t^2] E[\epsilon_{t-1}^2 \epsilon_t^2] = E_M[\sigma_{t-1}^2, \sigma_t^2],\end{aligned}\quad (2.47)$$

as $\epsilon_t \stackrel{iid}{\sim} N(0,1)$ and ϵ_t^2 and σ_t^2 are independent. Further note that the formula for $E_M[\sigma_{t-1}^2, \sigma_t^2]$ was already given in (2.17).

This completes the proof of the Theorem 2.2.3 \square

Theorem 2.2.4. *Let the diagonal elements of Ω be $\omega_{tt} = \text{Var}(Y_{t-1}^2, Y_t^2)$ and the lag 1 off diagonal elements be $\omega_{t-1,t} = \text{Cov}(Y_{t-1}^2, Y_t^2, Y_t^2, Y_{t+1}^2)$. The formulas for the variances are given by*

$$\omega_{tt} = \text{Var}[Y_{t-1}^2, Y_t^2] = \begin{cases} 9\sigma_{10}^4 \exp\left[2\gamma_1 \ln \sigma_{10}^2 + 2\sigma_0^2\right] - \psi_{12}^2 & \text{for } t=2 \\ 9 \exp\left[2\left(\gamma_1^{t-2} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2\right)\right] \\ + 2\sigma_0^2 \left((1+\gamma_1)^2 \sum_{i=0}^{t-3} \gamma_1^{2i} + 1\right) & \text{for } t=3, \dots, T, \end{cases} \quad (2.48)$$

and for $t=2, \dots, T$ and lag 1 covariances have the formula as

$$\begin{aligned}\omega_{t-1,t} &= \text{Cov}(Y_{t-1}^2, Y_t^2, Y_t^2, Y_{t+1}^2) = 3 \exp\left(\left[\gamma_1^{t-2} + 2\gamma_1^{t-1} + \gamma_1^t\right] \ln \sigma_{10}^2\right) \\ &+ \left[\frac{\sigma_0^2}{2} \left((1+\gamma_1)^4 \sum_{i=0}^{t-3} \gamma_1^{2i} + (2+\gamma_1)^2 + 1\right)\right] - \psi_{t-1,t} \psi_{t,t+1}\end{aligned}\quad (2.49)$$

Proof. First derive the formula for variances,

$$\text{Var}[Y_{t-1}^2, Y_t^2] = E[Y_{t-1}^4, Y_t^4] - E^2[Y_{t-1}^2, Y_t^2]$$

$$\begin{aligned}
&= E[\sigma_{t-1}^4 \sigma_t^4 \epsilon_{t-1}^4 \epsilon_t^4] - \psi_{t-1,t}^2 \\
&= E_M[\sigma_{t-1}^4 \sigma_t^4] E[\epsilon_{t-1}^4 \epsilon_t^4] - \psi_{t-1,t}^2 \quad t = 2, \dots, T. \quad (2.50)
\end{aligned}$$

Since $\epsilon_t \stackrel{iid}{\sim} N(0, 1)$ and ϵ_t^2 and σ_t^2 are independent. For convenience, we include the derivation of $E_M[\sigma_{t-1}^4 \sigma_t^4]$ in Appendix A. The expression is

$$\begin{aligned}
E_M[\sigma_{t-1}^4 \sigma_t^4] &= \exp \left[2 \left(\gamma_1^{t-2} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2 \right) \right. \\
&\quad \left. + 2 \sigma_0^2 \left((1 + \gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1 \right) \right] \quad (2.51)
\end{aligned}$$

with $E_M[\sigma_1^4 \sigma_2^4] = \sigma_{10}^4 E_M[\sigma_2^4]$, where $E_M[\sigma_2^4]$ is computed from (2.36). Now by using this expression from (A.2) into (2.50),

$$\begin{aligned}
\text{Var}[Y_{t-1}^2 Y_t^2] &= 9 E_M[\sigma_{t-1}^4 \sigma_t^4] - \psi_{t-1,t}^2 \\
&= 9 \exp \left[2 \left(\gamma_1^{t-2} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2 \right) \right. \\
&\quad \left. + 2 \sigma_0^2 \left((1 + \gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1 \right) \right] - \psi_{t-1,t}^2 \quad \text{for } t = 3, \dots, T, \quad (2.52)
\end{aligned}$$

whereas for $t=2$,

$$\begin{aligned}
\text{Var}[Y_1^2 Y_2^2] &= E[\sigma_1^4 \sigma_2^4] E[\epsilon_1^4 \epsilon_2^4] - \psi_{1,2}^2 = 9 E[\sigma_1^4] E[\sigma_2^4] - \psi_{1,2}^2 \\
&= 9 \sigma_{10}^4 E[\sigma_2^4] - \psi_{1,2}^2, \quad (2.53)
\end{aligned}$$

and the formula for $\psi_{t-1,t}$ is given in (2.45).

Next, the covariances, for $t=2, \dots, T$,

$$\text{Cov}(Y_{t-1}^2 Y_t^2, Y_t^2 Y_{t+1}^2) = E[Y_{t-1}^2 Y_t^4 Y_{t+1}^2] - E[Y_{t-1}^2 Y_t^2] E[Y_t^2 Y_{t+1}^2]$$

$$\begin{aligned}
&= E[\sigma_{t-1}^2 \sigma_t^4 \sigma_{t+1}^2 \epsilon_{t-1}^2 \epsilon_t^4 \epsilon_{t+1}^2] - \psi_{t-1,t} \psi_{t,t+1} \\
&= E_M[\sigma_{t-1}^2 \sigma_t^4 \sigma_{t+1}^2] E[\epsilon_{t-1}^2 \epsilon_t^4 \epsilon_{t+1}^2] - \psi_{t-1,t} \psi_{t,t+1} \\
&= 3 E_M[\sigma_{t-1}^2 \sigma_t^4 \sigma_{t+1}^2] - \psi_{t-1,t} \psi_{t,t+1} \quad (2.54)
\end{aligned}$$

as $\epsilon_t \stackrel{\text{ind}}{\sim} N(0, 1)$. For convenience, the lengthy derivation of $E_M[\sigma_{t-1}^2 \sigma_t^4 \sigma_{t+1}^2]$ is given in appendix A. The formula of $E_M[\sigma_{t-1}^2 \sigma_t^4 \sigma_{t+1}^2]$ is given below from the appendix. The formula is:

$$\begin{aligned}
E_M[\sigma_{t-1}^2 \sigma_t^4 \sigma_{t+1}^2] &= \exp\left(\left[\gamma_1^{t-2} + 2\gamma_1^{t-1} + \gamma_1^t\right] \log \sigma_{10}^2\right. \\
&\quad \left.+ \left[\frac{\sigma_{10}^2}{2} \left((1 + \gamma_1)^4 \sum_{i=0}^{t-3} \gamma_1^{2i} + (2 + \gamma_1)^2 + 1\right)\right]\right) \quad (2.55)
\end{aligned}$$

Hence the proof. \square

Computational Formula for γ_1 Estimate:

The GQL estimating equation (2.44) can be solve iteratively to obtain an estimate for γ_1 . The iterative equation has the form

$$\hat{\gamma}_1(r+1) = \hat{\gamma}_1(r) + \left[\left(\frac{\partial \psi'}{\partial \gamma_1} \Omega^{-1} \frac{\partial \psi}{\partial \gamma_1} \right)^{-1} \left(\frac{\partial \psi'}{\partial \gamma_1} \Omega^{-1} (\mathbf{v} - \psi) \right) \right]_{|r} \quad (2.56)$$

where $\frac{\partial \psi}{\partial \gamma_1}$ is the derivative of ψ w.r.t γ_1 . By (2.47) the derivative has the formula for $t=3, \dots, T$,

$$\begin{aligned}
\frac{\partial \psi_{t-1,t}}{\partial \gamma_1} &= \exp\left[\gamma_1^{t-2} \log \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2 + \frac{\sigma_{10}^2}{2} \left((1 + \gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1\right)\right] \\
&\quad \left[(t-2)\gamma_1^{t-3} \ln \sigma_{10}^2 + (t-1)\gamma_1^{t-2} \ln \sigma_{10}^2 \right. \\
&\quad \left. + \frac{\sigma_{10}^2}{2} \left(2(1 + \gamma_1) \sum_{l=0}^{t-3} \gamma_1^{2l} + (1 + \gamma_1)^2 \sum_{l=0}^{t-3} (2l) \gamma_1^{(2l-1)} \right) \right], \quad (2.57)
\end{aligned}$$

and $t=2 \quad \frac{\partial \psi_{1,2}}{\partial \gamma_1} = \sigma_{10}^2 \exp \left[\gamma_1 \ln \sigma_{10}^2 + \frac{\sigma_{\epsilon}^2}{2} \right] \log \sigma_{10}^2$. In (2.56) $\hat{\gamma}_1(r)$ is the value of γ_1 at t^{th} iteration, and $[\cdot]_{\hat{\gamma}_1(r)}$ is the value of the expression in the square bracket evaluated at $\gamma_1 = \hat{\gamma}_1(r)$. Let the final estimate of (2.56) is denoted by $\hat{\gamma}_{1,GQL}$.

2.3 A Working Generalized Quasi Likelihood (WGQL)

Method in Finite Time Series

Note that, the derivation for the formulas of the covariance matrices Σ (2.28) and Ω (2.44) is not complex but the computation for the inverse of these full covariance matrices is time consuming. To avoid such complexity, we will use a suitable simpler form for these matrices and construct the 'Working' GQL (WGQL) estimating equations for σ_{η}^2 in Sections 2.3.1 and for γ_1 in Section 2.3.2.

2.3.1 WGQL Estimating Equation for σ_{η}^2

To avoid the difficulty of obtaining Σ^{-1} for the GQL estimation of σ_{η}^2 , we pretend that Y_u^2 and Y_t^2 are uncorrelated, even though in reality they are correlated. Thus, we use

$$\Sigma_d = \text{diag}[\sigma_{11}, \dots, \sigma_{tt}, \dots, \sigma_{TT}] \quad (2.58)$$

to replace the Σ matrix in (2.28). It then follows that the Σ_d matrix based WGQL estimating equation is given

$$\frac{\partial \lambda'}{\partial \sigma_\eta^2} \Sigma_d^{-1} (\mathbf{u} - \lambda) = 0. \quad (2.59)$$

Similar to the equation (2.28) can be solve iteratively to obtain an estimate for σ_η^2 .

Let this final estimate of (2.59) is denoted by $\hat{\sigma}_{\eta, WGQL}^2$.

2.3.2 WGQL Estimating Equation for γ_1

Similar to the construction of the WGQL estimating equation for σ_η^2 , we construct the WGQL estimating equation for γ_1 by ignoring the covariances between $y_{u-1}^2 y_u^2$ and $y_{t-1}^2 y_t^2$. Thus we replace Ω in (2.44) with

$$\Omega_d = \text{diag}[\text{Var}(Y_1^2 Y_2^2), \dots, \text{Var}(Y_{t-1}^2 Y_t^2), \dots, \text{Var}(Y_{T-1}^2 Y_T^2)]. \quad (2.60)$$

The formulas for the unconditional variances of $(Y_{t-1}^2 Y_t^2)$ for $t = 2, \dots, T$ are given in (2.48).

We now write the WGQL estimating equation for γ_1 as

$$\frac{\partial \psi'}{\partial \gamma_1} \Omega_d^{-1} (\mathbf{v} - \psi) = 0, \quad (2.61)$$

which can be solved iteratively, that is, by solving (2.44) with $\Omega = \Omega_d$. Let the final estimate of γ_1 denoted by $\hat{\gamma}_{1, WGQL}$.

We remark that the WGQL estimating equations for σ_η^2 (2.59) for γ_1 (2.61) are similar to the well known weighted least square (WLS) equations for the corresponding parameters. These estimates will be consistent as the WGQL estimating equations are unbiased. However, the estimates may not be highly efficient, as for the construction of the WGQL estimating equations, we have replaced the true covariances Σ and Ω with their counter parts Σ_d and Ω_d , respectively.

2.4 Asymptotic Variances Comparison of the Estimators

2.4.1 Asymptotic Variances of the SMM Estimators

We provide the asymptotic variances of $\hat{\gamma}_{1,SMM}$ obtained from (2.24) as in the following lemma.

Lemma 2.4.1. *For $S_{20} = \sum_{t=T_0+2}^T y_{t-1}^2 y_t^2 / T - T_0 - 1$, the asymptotic variance of $\gamma_{1,SMM}$ is given by*

$$\lim_{T \rightarrow \infty} \text{Var}(\hat{\gamma}_{1,SMM}) = \left[\left(\frac{\partial g_{20}(\gamma_1, \sigma_\eta^2)}{\partial \gamma_1} \right)^{-2} \lim_{T_0 \rightarrow \infty} \text{Var}(S_{20}) \right], \quad (2.62)$$

where

$$\left(\frac{\partial g_{20}(\gamma_1, \sigma_\eta^2)}{\partial \gamma_1} \right) = \left(\frac{\sigma_\eta^2}{(1 - \gamma_1)^2} \right) g_{20}(\gamma_1, \sigma_\eta^2) = \delta_{20}, \quad (2.63)$$

and

$$\lim_{T \rightarrow \infty} \text{Var}(S_{20}) = \frac{1}{(T - T_0 - 1)^2} \left[(T - T_0 - 1) \xi_{20} + 2(T - T_0 - 2) \xi_{20}^* \right], \quad (2.64)$$

with

$$\xi_{20} = g_{20}^2(\gamma_1, \sigma_q^2) \left[9 g_{20}^2(\gamma_1, \sigma_q^2) - 1 \right] \quad \xi_{20}^* = 3 \exp \left[\sigma_q^2 \left(\frac{3 + \gamma_1}{1 - \gamma_1} \right) \right] - g_{20}^2(\gamma_1, \sigma_q^2).$$

Proof. The expression in (2.62) for the asymptotic variance of $\hat{\gamma}_{1SM}$ follows from (2.24). Note that the formula for the derivative is also available in (2.24). For convenience, we have re-expressed this derivation as in (2.63), where $g_{20}(\gamma_1, \sigma_q^2)$ is given in (2.23). Now it remains to show that, $\lim_{T_0 \rightarrow \infty} \text{Var}(S_{20})$ has the formula as in (2.64). For this purpose, we write

$$\begin{aligned} \lim_{T \rightarrow \infty} \text{Var}(S_{20}) &= \lim_{T \rightarrow \infty} \left[\frac{1}{(T - T_0 - 1)^2} \left(\sum_{t=T_0+2}^T \text{Var}(Y_{t-1}^2 Y_t^2) \right. \right. \\ &\quad \left. \left. + 2 \sum_{t=T_0+2}^{T-1} \text{Cov}(Y_{t-1}^2 Y_t^2, Y_t^2 Y_{t+1}^2) \right) \right]. \end{aligned} \quad (2.65)$$

Note that for a given t , the formula for $\text{Var}(Y_{t-1}^2 Y_t^2)$ is given in (2.48). When $T \rightarrow \infty$, we compute all necessary formulas for $t \rightarrow \infty$ case with $|\gamma_1| < 1$. Thus, we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Var}(Y_{t-1}^2 Y_t^2) &= 9 \exp \left(\frac{4\sigma_q^2}{1 - \gamma_1} \right) - \left[\exp \left(\frac{\sigma_q^2}{1 - \gamma_1} \right) \right]^2 \\ \xi_{20} &= g_{20}^2(\gamma_1, \sigma_q^2) \left[9 g_{20}^2(\gamma_1, \sigma_q^2) - 1 \right]. \end{aligned} \quad (2.66)$$

Similarly, for a given t , the formula for the $\text{Cov}(Y_{t-1}^2 Y_t^2, Y_t^2 Y_{t+1}^2)$ is given in (2.49).

Once again, for $T \rightarrow \infty$, i.e for $t \rightarrow \infty$, with $|\gamma_1| < 1$ we write

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Cov}(Y_{t-1}^2, Y_t^2, Y_t^2 Y_{t+1}^2) &= 3 \exp \left[\frac{\sigma_\eta^2}{2} \left(\frac{(1+\gamma_1)^4}{1-\gamma_1^2} + (2+\gamma_1)^2 + 1 \right) \right] - \left[\exp \left(\frac{\sigma_\eta^2}{1-\gamma_1} \right) \right]^2 \\ \xi_{20}^* &= 3 \exp \left[\sigma_\eta^2 \left(\frac{3+\gamma_1}{1-\gamma_1} \right) \right] - g_{20}^2(\gamma_1, \sigma_\eta^2) \end{aligned} \quad (2.67)$$

Hence the lemma. \square

Next, we provide the asymptotic variance of $\hat{\sigma}_{\eta, SMM}^2$ obtained from (2.13) in the following lemma.

Lemma 2.4.2. *By using (2.13), one may compute the asymptotic variance of the SMM estimator of $\hat{\sigma}_{\eta, SMM}^2$*

$$\lim_{T \rightarrow \infty} \text{Var}(\hat{\sigma}_{\eta, SMM}^2) = \left[\left(\frac{\partial f(s_{10})}{\partial s_{10}} \right)_{|s_{10}=g_{10}} \right]^2 \lim_{T \rightarrow \infty} \text{Var}(S_{10}), \quad (2.68)$$

where

$$\frac{\partial f(s_{10})}{\partial s_{10}} \Big|_{s_{10}=g_{10}} = 2(1-\gamma_1^2) S_{10}^{-1} \Big|_{s_{10}=g_{10}} = 2(1-\gamma_1^2) g_{10}^{-1}(\gamma_1, \sigma_\eta^2), \quad (2.69)$$

and

$$\lim_{T \rightarrow \infty} \text{Var}(S_{10}) = \frac{1}{(T-T_0)^2} \left[(T-T_0) \xi_{10} + \sum_{t=k+1, t=T_0+1}^T \xi_{10}^*(t-k, t) \right] \quad (2.70)$$

with

$$\xi_{10} = g_{10}^2(\gamma_1, \sigma_\eta^2) (3g_{10}^2(\gamma_1, \sigma_\eta^2) - 1)$$

and

$$\xi_{10}^*(t-k, t) = \exp \left(\sigma_\eta^2 \left[\frac{(1+\gamma_1^k)}{(1-\gamma_1^2)} \right] \right) - g_{10}^2(\gamma_1, \sigma_\eta^2).$$

Proof. The formula for $\text{Var}(\hat{\sigma}_{\text{SMM}}^2)$ follows from (2.13) along with derivatives in (2.69). Next, for the formula for $\text{Var}(S_{10})$, we write

$$\lim_{T \rightarrow \infty} \text{Var}(S_{10}) = \lim_{T \rightarrow \infty} \left[\frac{1}{(T - T_0)^2} \left(\sum_{t=T_0+1}^T \text{Var}(Y_t^2) + \sum_{t-k \neq t, t-k, t=T_0+1}^T \text{Cov}(Y_{t-k}^2, Y_t^2) \right) \right], \quad (2.71)$$

Note that for a given t , the expression for $\text{Var}(Y_t^2)$ is given in (2.34). When $T_0 \rightarrow \infty$, all necessary formulas are computed for $t \rightarrow \infty$ cases with $|\gamma_1| < 1$. Thus, we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Var}(Y_t^2) &= 3 \exp\left(\frac{2\sigma_\eta^2}{1-\gamma_1^2}\right) - \left[\exp\left(\frac{\sigma_\eta^2}{2} \left\{ \frac{1}{1-\gamma_1^2} \right\} \right) \right]^2 \\ \xi_{10} &= g_{10}^2(\gamma_1, \sigma_\eta^2) \left(3g_{10}^2(\gamma_1, \sigma_\eta^2) - 1 \right). \end{aligned}$$

To compute the limiting value for the second part in (2.71), we recall the formula for $\text{Cov}(Y_{t-k}^2, Y_t^2)$ from (2.33). Now, for $T_0 \rightarrow \infty$, i.e for $t \rightarrow \infty$ with $|\gamma_1| < 1$, by (2.33), for $t-k < t$, and $k=1, \dots, t-1$, we write

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Cov}(Y_{t-k}^2, Y_t^2) &= \exp\left(\sigma_\eta^2 \left[\frac{(1+\gamma_1^k)}{(1-\gamma_1^k)} \right] \right) - \left[\exp\left(\frac{\sigma_\eta^2}{2(1-\gamma_1^2)}\right) \right]^2 \\ \xi_{10}^*(t-k, t) &= \exp\left(\sigma_\eta^2 \left[\frac{(1+\gamma_1^k)}{(1-\gamma_1^k)} \right] \right) - g_{10}^2(\gamma_1, \sigma_\eta^2). \end{aligned} \quad (2.72)$$

This completes the proof of the lemma. \square

2.4.2 Asymptotic variances of the WGQL Estimators

In order to obtain the asymptotic variances of the WGQL estimators of γ_1 and σ_η^2 , we first provide their exact variances expressions in the following lemma.

Lemma 2.4.3. *The WGQL estimators for γ_1 and σ_v^2 obtained by solving (2.61) and (2.59), have the exact variances given by*

$$\text{Var}(\hat{\gamma}_{1WGQL}) = \left[\frac{\partial \psi'}{\partial \gamma_1} \Omega_d^{-1} \frac{\partial \psi}{\partial \gamma_1} \right]^{-1} \left[\frac{\partial \psi'}{\partial \gamma_1} \Omega_d^{-1} \Omega \Omega_d^{-1} \frac{\partial \psi}{\partial \gamma_1} \right] \left[\frac{\partial \psi'}{\partial \gamma_1} \Omega_d^{-1} \frac{\partial \psi}{\partial \gamma_1} \right]^{-1} \quad (2.73)$$

and

$$\text{Var}(\hat{\sigma}_v^2) = \left[\frac{\partial \lambda'}{\partial \sigma_v^2} \Sigma_d^{-1} \frac{\partial \lambda}{\partial \sigma_v^2} \right]^{-1} \left[\frac{\partial \lambda'}{\partial \sigma_v^2} \Sigma_d^{-1} \Sigma \Sigma_d^{-1} \frac{\partial \lambda}{\partial \sigma_v^2} \right] \left[\frac{\partial \lambda'}{\partial \sigma_v^2} \Sigma_d^{-1} \frac{\partial \lambda}{\partial \sigma_v^2} \right]^{-1} \quad (2.74)$$

respectively.

Proof. The lemma is obvious from the estimating equation (2.61) and (2.59). This is because, under the true model, $E[(\mathbf{u} - \lambda)(\mathbf{u} - \lambda)'] = \Sigma$ and $E[(\mathbf{v} - \psi)(\mathbf{v} - \psi)'] = \Omega$. \square

Lemma 2.4.4. [Asymptotic Variances] *For $\lim T_0 \rightarrow \infty$, the asymptotic variances of $\hat{\gamma}_{1WGQL}$ and $\hat{\sigma}_v^2$ are given by*

$$\lim_{T \rightarrow \infty} \text{Var}(\hat{\gamma}_{1WGQL}) = \left[\frac{\partial \psi'_0}{\partial \gamma_1} \Omega_{\infty}^{-1} \frac{\partial \psi_0}{\partial \gamma_1} \right]^{-1} \left[\frac{\partial \psi'_0}{\partial \gamma_1} \Omega_{\infty}^{-1} \Omega_0 \Omega_{\infty}^{-1} \frac{\partial \psi_0}{\partial \gamma_1} \right] \left[\frac{\partial \psi'_0}{\partial \gamma_1} \Omega_{\infty}^{-1} \frac{\partial \psi_0}{\partial \gamma_1} \right]^{-1} \quad (2.75)$$

and

$$\lim_{T \rightarrow \infty} \text{Var}(\hat{\sigma}_v^2) = \left[\frac{\partial \lambda'_0}{\partial \sigma_v^2} \Sigma_{\infty}^{-1} \frac{\partial \lambda_0}{\partial \sigma_v^2} \right]^{-1} \left[\frac{\partial \lambda'_0}{\partial \sigma_v^2} \Sigma_{\infty}^{-1} \Sigma_0 \Sigma_{\infty}^{-1} \frac{\partial \lambda_0}{\partial \sigma_v^2} \right] \left[\frac{\partial \lambda'_0}{\partial \sigma_v^2} \Sigma_{\infty}^{-1} \frac{\partial \lambda_0}{\partial \sigma_v^2} \right]^{-1} \quad (2.76)$$

where

$$\frac{\partial \psi'_0}{\partial \gamma_1} = \lim_{T \rightarrow \infty} \frac{\partial \psi'}{\partial \gamma_1} \quad \quad \quad \frac{\partial \lambda'_0}{\partial \sigma_v^2} = \lim_{T \rightarrow \infty} \frac{\partial \lambda'}{\partial \sigma_v^2}$$

$$\Omega_{d0} = \lim_{T \rightarrow \infty} \Omega_d, \quad \Sigma_{d0} = \lim_{T \rightarrow \infty} \Sigma_d, \quad \Sigma_0 = \lim_{T \rightarrow \infty} \Sigma, \quad \Omega_0 = \lim_{T \rightarrow \infty} \Omega. \quad (2.77)$$

where $\frac{\partial \psi}{\partial \gamma_1}, \Omega, \frac{\partial \lambda}{\partial \sigma_\eta^2}$ and Σ are as in the Lemma 2.4.3.

Proof. To compute the limiting variances from the exact variances in Lemma 2.4.3, we simply compute the limiting vectors and matrices componentwise. The formulas for the components of $\psi = [\psi_{12}, \dots, \psi_{t-1,t}, \dots, \psi_{T-1,T}]'$ and $\lambda = [\lambda_1, \dots, \lambda_t, \dots, \lambda_T]'$ are given in (2.42) and (2.31) respectively. We use these formulas and we obtain the derivatives

$$\lim_{t \rightarrow \infty} \frac{\partial \psi_{t-1,t}}{\partial \gamma_1} = \left(\frac{\sigma_\eta^2}{(1 - \gamma_1)^2} \right) g_{20}(\gamma_1, \sigma_\eta^2) = \delta_{20} \quad (2.78)$$

and

$$\lim_{t \rightarrow \infty} \frac{\partial \lambda_t}{\partial \sigma_\eta^2} = \left(\frac{1}{2(1 - \gamma_1^2)} \right) g_{10}(\gamma_1, \sigma_\eta^2) = \delta_{10} \text{ (say)}. \quad (2.79)$$

Recall from (2.58) that

$$\Sigma_d = \text{diag}[\text{Var}(Y_1^2), \dots, \text{Var}(Y_t^2), \dots, \text{Var}(Y_T^2)]$$

and from (2.60) that

$$\Omega_d = \text{diag}[\text{Var}(Y_1^2 Y_2^2), \dots, \text{Var}(Y_{t-1}^2 Y_t^2), \dots, \text{Var}(Y_{T-1}^2 Y_T^2)].$$

We obtain Σ_{d0} and Ω_{d0} by computing the limiting values of the components of Σ_d and Ω_d . These limiting values are

$$\lim_{t \rightarrow \infty} \text{Var}(Y_{t-1}^2 Y_t^2) = \xi_{20} \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{Var}(Y_t^2) = \xi_{10}. \quad (2.80)$$

where ξ_{20} and ξ_{10} are given by is in (2.65) and (2.71) respectively.

Further note that, the formulas for the diagonal and off diagonal elements of Σ_0 are given in (2.80) and (2.72) respectively. Also the diagonal and off diagonal elements of Ω are available in (2.80) and (2.67). \square

Note that, we have introduced a simpler MM (SMM) approach in Section 2.1 for the estimation of the volatility parameters γ_1 and σ_η^2 . The asymptotic variances of these SMM estimators are given in Section 2.4.1. Also we have discussed the WGQL estimation approach in Section 2.2 and the asymptotic variances of the WGQL estimators are given in Section 2.4.2. In Section 2.5, we will conduct an empirical study to examine the asymptotic performances of the proposed SMM and WGQL estimators. In the empirical study, we will also include the QML approach for the asymptotic variance comparison. The reason for this inclusion is that the QML approximation is computationally manageable, whereas the GMM approach is extremely cumbersome as it is developed based on large number of unbiased moment functions. For the purpose, in Section 2.4.3.1 we provide brief discussion on the QML approach and in Section 2.4.3.2 we given the formulas for the asymptotic covariance matrix of the QML estimators.

2.4.3 QML Estimators and Their Asymptotic variances

2.4.3.1 QML Estimation

Recall from (1.22) that

$$\begin{aligned} Z_t = \log y_t^2 &= E[\ln \epsilon_t^2] + \ln \sigma_t^2 + u_t \\ &\equiv \kappa_1 + \ln \sigma_t^2 + u_t \quad t = 1, \dots, T, \end{aligned} \quad (2.81)$$

where $\epsilon_t \stackrel{iid}{\sim} N(0, 1)$ and $\kappa_1 = -1.27$. In (2.81), u_t follows the log chi-square distribution with mean zero and variance $\kappa_2 = \pi^2/2$ [Abramovitz and Stegun (1970, p. 943)]. As we discussed in Section 1.2.2 that the multi-dimensional integration in (1.23) is extremely difficult, many authors such as Ruiz (1994) and Harvey et al (1994), have approximated the distribution by pretending that $\mathbf{Z} = (Z_1, Z_2, \dots, Z_T)'$ follows a MVN (multivariate normal) distribution with true mean vector and true covariance matrix under the model (2.81). Let $\mathbf{m} = (m_1, \dots, m_t, \dots, m_T)' = E[\mathbf{Z}]$ and $\mathbf{V} = \text{Cov}(\mathbf{Z}) = (v_{st})$ be the true mean and the covariance matrix of the response vector \mathbf{Z} .

Lemma 2.4.5. *Under the model (2.81), the expectation of z_t is given by*

$$m_t = \begin{cases} -1.27 + \ln \sigma_{10}^2 & \text{for } t = 1 \\ -1.27 + \gamma_1^{t-1} \ln \sigma_{10}^2 & \text{for } t=2, \dots, T \end{cases}$$

Proof. We can write Z_t by using the recurrence and (1.2) relationship as

$$Z_t = \kappa_1 + \gamma_1 h_{t-1} + \eta_t + u_t$$

$$\begin{aligned}
&= \kappa_1 + \gamma_1^{t-1} \ln \sigma_{10}^2 + \gamma_1^{t-2} \eta_2 + \gamma_1^{t-3} \eta_3 + \dots + \gamma_1 \eta_{t-1} + \eta_t + u_t \\
&= \kappa_1 + \gamma_1^{t-1} \ln \sigma_{10}^2 + \sum_{i=2}^t \gamma_1^{t-i} \eta_i + u_t,
\end{aligned} \tag{2.82}$$

for $t=2, \dots, T$.

Since $E[u_t] = 0$ and $E[\eta_k] = 0$, for $\sigma_1^2 = \sigma_{10}^2$, it now follows that

$$E[Z_1] = \kappa_1 + \ln \sigma_{10}^2, \tag{2.83}$$

and

$$E(Z_t) = \kappa_1 + \gamma_1^{t-1} \ln \sigma_{10}^2 = m_t \quad \text{for } t = 2, \dots, T, \tag{2.84}$$

where $\kappa_1 = -1.27$. Hence the lemma. \square

Lemma 2.4.6. *The elements of the covariance matrix of $\mathbf{Z} = (Z_1, Z_2, \dots, Z_T)'$ have the formulas:*

$$v_{t-k,t} = \begin{cases} \kappa_2 & \text{for } k=0, t=1 \\ 0 & \text{for } t-k=1, t=2, \dots, T \\ \sigma_0^2 \sum_{i=2}^t \gamma_1^{2(t-i)} + \kappa_2 & \text{for } k=0, t=2, \dots, T \\ \sigma_0^2 \sum_{i=2}^{t-k} \gamma_1^{2(t-i)-k} & \text{for } k=1, \dots, t-2 \text{ and } t=3, \dots, T \end{cases} \tag{2.85}$$

and $v_{11} = v_{21}$.

Proof. The computation of $\text{Var}(Z_t)$ is straightforward from (2.82). i.e.

$$\begin{aligned}
\text{Var}(Z_t) &= \sum_{i=2}^t \gamma_1^{2(t-i)} \text{Var}(\eta_i) + \text{Var}(u_t) \\
&= \sigma_0^2 \sum_{i=2}^t \gamma_1^{2(t-i)} + \pi^2/2 = v_{tt} \quad t = 2, \dots, T,
\end{aligned}$$

with $\text{Var}(Z_1) = \kappa_2 = \pi^2/2$.

We now derive the formula for the covariance between Z_1 and Z_t . Recall that

$$\begin{aligned} z_1 &= \kappa_0 + h_1 + u_1 \\ z_t &= \kappa_0 + \gamma_1^{t-1} \ln \sigma_{10}^2 + \sum_{i=2}^t \gamma_1^{(t-i)} \eta_i + u_t \quad t = 2, \dots, T. \end{aligned}$$

For calculation for the covariances, we simply write

$$\text{Cov}(Z_1, Z_t) = E[(Z_1 - E[Z_1])(Z_t - E[Z_t])] \quad (2.86)$$

By using $E[Z_1]$ and $E[Z_t]$ from (2.83) and (2.84) in (2.86), after a simple algebra, we get

$$\begin{aligned} \text{Cov}(Z_1, Z_t) &= E\left[u_1 \left(\sum_{i=2}^t \gamma_1^{(t-i)} \eta_i + u_t\right)\right] \\ &= E[u_1]E\left[\sum_{i=2}^t \gamma_1^{(t-i)} \eta_i\right] + E[u_1 u_t] \\ &= 0. \end{aligned} \quad (2.87)$$

This is because $u_t \stackrel{iid}{\sim} \log \chi^2(0, \kappa_2)$ under the model (2.81).

Next, for lag k ($=1, \dots, t-2$) and $t=3, \dots, T$, we write

$$Z_{t-k} = \kappa_0 + \gamma_1^{t-k-1} \ln \sigma_{10}^2 + \sum_{i=2}^{t-k} \gamma_1^{(t-k-i)} \eta_i + u_{t-k}. \quad (2.88)$$

and

$$Z_t = \kappa_0 + \gamma_1^{t-1} \ln \sigma_{10}^2 + \left[\sum_{i=2}^{t-k} \gamma_1^{(t-i)} \eta_i + \sum_{i=t-k+1}^t \gamma_1^{(t-i)} \eta_i \right] + u_t. \quad (2.89)$$

It is clear from (2.88) and (2.89), we write the may write the formula for the auto-covariance between Y_{t-k} and Y_t as

$$\begin{aligned} \text{Cov}(Z_{t-k}, Z_t) &= E[(Z_{t-k} - E[Z_t])(Z_t - E[Z_t])] \\ &= \sigma_\eta^2 \sum_{i=2}^{t-k} \gamma_1^{2(t-i)-k} = v_{t-k,t} \quad k = 1, \dots, t-2 \text{ and } t = 3, \dots, T \end{aligned} \quad (2.90)$$

□

Next, by using the true mean and the true covariance matrix, and by pretending that \mathbf{Z} follows a multivariate normal distribution, we may write an approximate log likelihood function given by

$$\log L_Q^* = c_0 - \frac{1}{2} \log |\mathbf{V}| - \frac{1}{2} [(\mathbf{Z} - \mathbf{m})' \mathbf{V}^{-1} (\mathbf{Z} - \mathbf{m})], \quad (2.91)$$

[Shephard, (1996 eqn:1.17)]. This approximate likelihood L_Q^* in (2.91) is referred to as the quasilielihood (QL). It then follows that the quasi maximum likelihood (QML) estimates for γ_1 and σ_η^2 can be obtained by solving

$$\begin{aligned} \frac{\partial \log L_Q^*}{\partial \gamma_1} &= -\frac{1}{2} \frac{\partial \log |\mathbf{V}|}{\partial \gamma_1} - \frac{\partial (\mathbf{Z} - \mathbf{m})'}{\partial \gamma_1} \mathbf{V}^{-1} (\mathbf{Z} - \mathbf{m}) - \frac{1}{2} (\mathbf{Z} - \mathbf{m})' \frac{\partial \mathbf{V}^{-1}}{\partial \gamma_1} (\mathbf{Z} - \mathbf{m}) \\ &= -\frac{1}{2} \text{trace}[\mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \gamma_1}] + d' \mathbf{V}^{-1} (\mathbf{Z} - \mathbf{m}) + \frac{1}{2} (\mathbf{Z} - \mathbf{m})' \mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \gamma_1} \mathbf{V}^{-1} (\mathbf{Z} - \mathbf{m}) \\ &= 0 \end{aligned} \quad (2.92)$$

where $d' = [0, \hat{g}_2(\gamma_1), \dots, \hat{g}_T(\gamma_1)]$, with $\frac{\partial m_t}{\partial \gamma_1} = (t-1) \gamma_1^{t-2} \ln \sigma_{10}^2 = \hat{g}_t(\gamma_1) \quad t = 2, \dots, T$,

and

$$\begin{aligned}
 \frac{\partial \log L_Q^*}{\partial \sigma_\eta^2} &= -\frac{1}{2} \frac{\partial \log |\mathbf{V}|}{\partial \sigma_\eta^2} - \frac{1}{2} (\mathbf{Z} - \mathbf{m})' \frac{\partial \mathbf{V}^{-1}}{\partial \sigma_\eta^2} (\mathbf{Z} - \mathbf{m}) \\
 &= -\frac{1}{2} \text{trace}[\mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \sigma_\eta^2}] + \frac{1}{2} (\mathbf{Z} - \mathbf{m})' \mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \sigma_\eta^2} \mathbf{V}^{-1} (\mathbf{Z} - \mathbf{m}) \\
 &= 0
 \end{aligned} \tag{2.93}$$

respectively. The detail derivatives of $\frac{\partial \mathbf{V}}{\partial \gamma_1}$ and $\frac{\partial \mathbf{V}}{\partial \sigma_\eta^2}$ are given in Appendix B. Let the final QML estimates from (2.92) and (2.93) be denoted by $\hat{\gamma}_{1,QML}$ and $\hat{\sigma}_{\eta,QML}^2$, respectively.

Note that, the true distribution of u_t , namely $(\log \chi^2)$ distribution is extremely left skewed. This implies that conditional on $\ln \sigma_t^2$, z_t follows the $\log \chi^2$ distribution. Consequently, the aforementioned normality based QML approximation can be inefficient. We will examine this efficiency issue empirically in Section 2.5 by using the asymptotic variance formulas for the QML estimators given in Section 2.4.3.2.

2.4.3.2 Asymptotic variances of the QML Estimators

For $\alpha = (\gamma_1, \sigma_\eta^2)'$, for any T , small or large it follows from (2.92) and (2.93) that the asymptotic covariance matrix of the QML estimator of α is given by the Fisher information matrix defined as

$$\text{Cov}(\hat{\alpha}_{QML}) = - \left(\begin{array}{cc} E \left[\frac{\partial^2 \log L_Q^*}{\partial \gamma_1^2} \right] & E \left[\frac{\partial^2 \log L_Q^*}{\partial \gamma_1 \partial \sigma_\eta^2} \right] \\ E \left[\frac{\partial^2 \log L_Q^*}{\partial (\sigma_\eta^2)^2} \right] & \end{array} \right)^{-1}, \tag{2.94}$$

where

$$\begin{aligned}
 E \left[\frac{\partial^2 \log L_Q^*}{\partial \gamma_1^2} \right] &= -\frac{1}{2} \frac{\partial [\text{trace}(\mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \gamma_1})]}{\partial \gamma_1} - d' \mathbf{V}^{-1} d - \frac{1}{2} \text{trace} \left[\frac{\partial^2 \mathbf{V}^{-1}}{\partial \gamma_1^2} \mathbf{V} \right] \\
 E \left[\frac{\partial^2 \log L_Q^*}{\partial \gamma_1 \partial \sigma_\eta^2} \right] &= -\frac{1}{2} \frac{\partial [\text{trace}(\mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \gamma_1})]}{\partial \sigma_\eta^2} - \frac{1}{2} \text{trace} \left[\frac{\partial (\frac{\partial \mathbf{V}^{-1}}{\partial \gamma_1})}{\partial \sigma_\eta^2} \mathbf{V} \right] \\
 E \left[\frac{\partial^2 \log L_Q^*}{\partial (\sigma_\eta^2)^2} \right] &= -\frac{1}{2} \frac{\partial [\text{trace}(\mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \sigma_\eta^2})]}{\partial \sigma_\eta^2} - \frac{1}{2} \text{trace} \left[\frac{\partial (\frac{\partial \mathbf{V}^{-1}}{\partial \sigma_\eta^2})}{\partial \sigma_\eta^2} \mathbf{V} \right] \quad (2.95)
 \end{aligned}$$

with $\mathbf{V} = \text{Cov}(\mathbf{Z})$, where $\mathbf{Z} = (Z_1, \dots, Z_t, \dots, Z_T)'$ with $Z_t = \log y_t^2$. Further, the derivatives for (2.95) have the formulas as shown in the Appendix B.

Note that for the empirical study in the following section, we will compute the variances and covariances in (2.94) for the case when $t \rightarrow \infty$. These variances and covariances, for convenience, we referred to as the asymptotic variances and covariances.

2.5 Asymptotic Variance Comparison: An Empirical Study

Recall that many authors such as Ruiz (1994) and Anderson and Sorenson (1997) have compared the asymptotic variances of the GMM estimators with that of the QML estimators for the estimation of the standard volatility parameters γ_1 and σ_η^2 . But as it was argued in Section 1.2.1 that finding the GMM estimates by solving (1.19) is quite cumbersome, because of the fact that it requires an arbitrary large number of

unbiased moment functions (Anderson and Sorenson (1996)). Consequently, we have avoided the formulation for GMM estimation approach but concentrated on SMM, WGQL estimation in Section 2.1 and 2.2. In the last section, we have shown how to compute the variances of the QML estimates. We now examine the relative efficiency performances of the proposed SMM and WGQL estimators with the corresponding QML estimators.

For convenience, similar to the existing studies [Ruiz (1994) and Anderson and Sorenson (1997), Broto and Ruiz (2004)] we consider the case $\gamma_0 = 0$ and select the values for the parameters of interest as follows;

$$\begin{aligned}\gamma_1 &= 0.25, \quad \text{and} \quad 0.5 \\ \sigma_\eta^2 &= 0.25, 0.5 \quad \text{and} \quad 1.0.\end{aligned}$$

For the computation of the asymptotic variances, we have chosen the time series with length $T = 1000, 2000$ and 3000 . The asymptotic variances of the proposed SMM, WGQL and the QML estimators for γ_1 are computed by (2.62), (2.75) and (2.94) respectively. Similarly, the asymptotic variances of the proposed SMM, WGQL and the QML estimators for σ_η^2 are computed by (2.68), (2.76) and (2.94) respectively. These variances for various selected values of the parameters are shown in Table 2.1.

Table 2.1: Asymptotic variance comparison of SMM, WGQL and QML estimators with selected parameter values

σ_η^2	γ_1	Method	Parameters	Time Series Length (T)		
				1000	2000	3000
0.25	0.25	SMM	γ_1	0.1198	0.0598	0.0399
			σ_η^2	0.0109	0.0054	0.0036
		WGQL	γ_1	0.0835	0.0418	0.0279
			σ_η^2	0.0103	0.0051	0.0034
		QML	γ_1	0.4163	0.4143	0.4132
			σ_η^2	0.1350	0.1328	0.1322
0.25	0.5	SMM	γ_1	0.0342	0.0171	0.0114
			σ_η^2	0.0088	0.0044	0.0029
		WGQL	γ_1	0.0236	0.0118	0.0078
			σ_η^2	0.0072	0.0036	0.0024
		QML	γ_1	0.2649	0.2662	0.2660
			σ_η^2	0.1480	0.1471	0.1467
0.5	0.25	SMM	γ_1	0.0569	0.0285	0.0190
			σ_η^2	0.0158	0.0079	0.0053
		WGQL	γ_1	0.0421	0.0210	0.0140
			σ_η^2	0.0145	0.0072	0.0048
		QML	γ_1	0.3889	0.3882	0.3865
			σ_η^2	0.5254	0.5210	0.5180

(Table 2.1 Contd....)

σ_v^2	γ_1	Method	Parameters	Time Series Length (T)		
				1000	2000	3000
0.5	0.5	SMM	γ_1	0.0226	0.0113	0.0075
			σ_v^2	0.0143	0.0071	0.0048
		WGQL	γ_1	0.0165	0.0082	0.0055
			σ_v^2	0.0109	0.0055	0.0036
		QML	γ_1	0.2252	0.2252	0.2265
			σ_v^2	0.5528	0.5508	0.5500
1.0	0.25	SMM	γ_1	0.0501	0.0251	0.0167
			σ_v^2	0.0299	0.0150	0.0100
		WGQL	γ_1	0.0408	0.0204	0.0136
			σ_v^2	0.0272	0.0136	0.0091
		QML	γ_1	0.3477	0.3463	0.3448
			σ_v^2	2.0489	2.0336	2.0447
1.0	0.5	SMM	γ_1	0.0381	0.0190	0.0127
			σ_v^2	0.0310	0.0155	0.0103
		WGQL	γ_1	0.0618	0.0154	0.0102
			σ_v^2	0.0234	0.0117	0.0078
		QML	γ_1	0.1775	0.1768	0.1785
			σ_v^2	2.0621	2.0611	2.0642

It is clear from the results in Table 2.1 that SMM and WGQL approaches appear to produce estimates for γ_1 and σ_η^2 with smaller variances as compared to the QML approach in all cases. For example, when $T = 2000$, $\sigma_\eta^2 = 0.5$ and $\gamma_1 = 0.5$, the SMM approach is

$$Eff(\hat{\gamma}_1) = \frac{Var(\hat{\gamma}_{1,QML})}{Var(\hat{\gamma}_{1,SMM})} = \frac{0.2252}{0.0113} = 19.93$$

times more efficient than the QML in estimate of γ_1 . Similarly the WGQL approach is 100.15 times more efficient in estimating σ_η^2 as compared to QML approach.

Note that as it was argued earlier that the so-called GMM (1.19) is cumbersome (which makes it impractical), we did not include such a complex arbitrary technique in our comparison [see also Anderson et.al, 1999, section 1, p. 63-64]. Nevertheless, by comparing the existing asymptotic variances for the GMM and QML estimates from Anderson and Sorenson 1997, Table 1, (p. 401) with those of the proposed SMM estimates, for example, one may understand the relative performances of the proposed simpler MM (SMM) approach to the GMM and QML approaches. To be specific, consider the estimation for $\gamma_1 = 0.7$ and $\sigma_\eta^2 = 1.0$. For these parameter values, the results from Table 1 in Anderson and Sorenson (1997) show that for the case with $T = 75,000$, the GMM is $2.31^2/2.03^2 = 1.30$ and $10.74^2/6.44^2 = 2.78$ times more efficient than the QML in estimating γ_1 and σ_η^2 , respectively. For the same parameter combination, we, however, find that the proposed SMM approach produces 0.0010 and zero asymptotic variances for γ_1 and σ_η^2 estimates respectively.

Thus, the proposed SMM approach is $2.03^2/0.0010 = 4120.9$ times more efficient than the GMM approach in estimating γ_1 and it is much more efficient than the GMM approach in estimating σ_η^2 .

For certain combination of parameter values such as $\gamma_1 = 0.97$ and $\sigma_\eta^2 = 0.04$, the results from Table 1 in Anderson and Sorenson (1997) show the QML is more efficient than the GMM approach. Now, for the same parameter values, our SMM approach gives zero asymptotic variances for both parameters. Thus, the proposed SMM approach is highly efficient than the QML approach. These comparative results, therefore, show that the proposed SMM is better than the existing QML and/or GMM approaches. Note that, the proposed SMM is much more simpler than the existing QML and GMM approaches.

When WGQL is compared to the SMM approach, they are found to be performing almost the same, the WGQL being slightly better. For example, when $T=1000$, $\sigma_\eta^2 = 1.0$ and $\gamma_1 = 0.25$, the WGQL approach is

$$Eff(\hat{\gamma}_1) = \frac{Var(\hat{\gamma}_{1,SMM})}{Var(\hat{\gamma}_{1,WGQL})} = \frac{0.0501}{0.0408} = 1.23$$

times more efficient than the SMM in estimate of γ_1 . Similarly the WGQL approach is 1.10 times more efficient in estimating σ_η^2 as compared to SMM approach. Note that a complete GQL approach, as opposed to the WGQL approach, could be highly efficient than the SMM approach. But for computational simplicity we have considered the WGQL approach which, as expected produced slightly improved estimates over the

SMM approach.

Chapter 3

Small and Large Sample

Estimation Performance of the

Proposed SMM and WGQL

Approaches: A Simulation Study

In Section 2.5, it was demonstrated through Table 2.1 that the proposed SMM and WGQL approaches are asymptotically more efficient than the existing QML approach. Also it was argued that the existing GMM approach is cumbersome and can be less efficient as compared to the proposed SMM approach. Note that the proposed approaches, the SMM approach in particular, are much simpler than the QML and

GMM approaches. In this chapter, we examine both small and large sample estimation performances of the proposed SMM and WGQL approaches through a simulation study. For the purpose, we choose small as well as large values for T . We consider small time series with length (T) up to 500. Any series with length more than 500 is consider to be a large series, and we choose the values of T as large as $T = 10,000$. Note that these values of T are chosen to indicate that unlike the existing GMM approach (where length of time series requires to be infinitely large such as $T = 10,000$ or 15,000, ..., and so on) the proposed approaches produces good estimates based on a practically reasonable length of the time series.

3.1 Small Sample Case

In the small sample case the initial variance σ_1^2 will have an effect on the estimation of the main volatility parameters, as expected. Since $\log \sigma_1^2$ is assumed to have the normal distribution with mean $\gamma_0/(1 - \gamma_1)$ as shown in (1.3), one may choose a value for σ_1^2 such that $\log \sigma_1^2$ is close to its mean value 0. In the present simulation study, we have used, for example, $\sigma_1^2 = \sigma_{10}^2 = 1.25$ for the SMM and the WGQL approaches. Now, to examine the small sample estimation performance for γ_1 and σ_η^2 by the SMM approach, we solve the SMM estimating equation (2.14) for γ_1 , and (2.3) for σ_η^2 , iteratively. The simulated mean (SM) along with simulated standard errors (SSE) (also simulated mean square error (SMSE)) for the SMM estimates

based on 1000 simulations are reported in Table 3.1 for various small time series with length up to 500. For the estimation of γ_1 and σ_η^2 by the WGQL approach, we solve the WGQL estimating equations (2.44) for γ_1 , and (2.59) for σ_η^2 , iteratively. The simulated estimates and their standard errors for the WGQL estimates are given in Table 3.2.

Table 3.1: Simulated mean (SM), simulated standard error (SSE) and simulated mean square error (SMSE) of the SMM estimates based on small time series with $T = 100, 200, 300$ & 500 for selected parameters values by using 1000 simulations.

σ_η^2	γ_1	Parameters	Quantity	Time Series Length (T)			
				100	200	300	500
0.25	0.25	γ_1	SM	0.2818	0.2517	0.2804	0.2421
			SSE	0.5149	0.4964	0.4700	0.4222
			MSE	0.2661	0.2464	0.2218	0.1783
		σ_η^2	SM	0.2276	0.2162	0.2148	0.2195
			SSE	0.2543	0.2021	0.1880	0.1547
			MSE	0.0652	0.0420	0.0366	0.0249
0.25	0.5	γ_1	SM	0.3178	0.3310	0.3992	0.3852
			SSE	0.5026	0.4663	0.4301	0.3643
			MSE	0.2858	0.2460	0.1952	0.1459
		σ_η^2	SM	0.2710	0.2605	0.2501	0.2559
			SSE	0.2971	0.2329	0.2174	0.1760
			MSE	0.0887	0.0543	0.0472	0.0310
0.5	0.25	γ_1	SM	0.2345	0.2102	0.2325	0.2406
			SSE	0.4759	0.4242	0.3737	0.3179
			MSE	0.2267	0.1816	0.1400	0.1035
		σ_η^2	SM	0.4120	0.4300	0.4376	0.4565
			SSE	0.3692	0.2927	0.2637	0.2066
			MSE	0.1440	0.0906	0.0734	0.0446

(Table 3.1 Contd....)

σ_η^2	γ_1	Parameters	Quantity	Time Series Length (T)			
				100	200	300	500
0.5	0.5	γ_1	SM	0.3351	0.3280	0.3815	0.3910
			SSE	0.4305	0.3909	0.3417	0.2779
			MSE	0.2125	0.1824	0.1308	0.0891
		σ_η^2	SM	0.4892	0.5099	0.4989	0.5128
			SSE	0.4370	0.3473	0.3151	0.2533
			MSE	0.1911	0.1207	0.0993	0.0643
1.0	0.25	γ_1	SM	0.1685	0.1610	0.1822	0.1864
			SSE	0.4097	0.3361	0.2929	0.2467
			MSE	0.1745	0.1209	0.0904	0.0649
		σ_η^2	SM	0.8239	0.8925	0.9185	0.9460
			SSE	0.5565	0.4310	0.3800	0.2997
			MSE	0.3407	0.1973	0.1510	0.0927
1.0	0.5	γ_1	SM	0.2912	0.3082	0.3656	0.3837
			SSE	0.3819	0.3252	0.2958	0.2572
			MSE	0.1894	0.1426	0.1056	0.0797
		σ_η^2	SM	0.9850	1.0268	1.0098	1.0240
			SSE	0.6897	0.5441	0.5106	0.4288
			MSE	0.4759	0.2968	0.2608	0.1844

Table 3.2: Simulated mean (SM), simulated standard error (SSE) and simulated mean square error (SMSE) of the WGQL estimates based on small time series with $T = 100, 200, 300$ & 500 for selected parameters values by using 1000 simulations.

σ_η^2	γ_1	Parameters	Quantity	Time Series Length (T)			
				100	200	300	500
0.25	0.25	γ_1	SM	0.3536	0.3436	0.3628	0.2894
			SSE	0.4569	0.4478	0.4286	0.4048
			SMSE	0.2195	0.2093	0.1964	0.1654
		σ_η^2	SM	0.2251	0.2223	0.2091	0.2178
			SSE	0.4739	0.3621	0.1996	0.1748
			SMSE	0.2252	0.1319	0.0415	0.0316
0.25	0.5	γ_1	SM	0.3685	0.3885	0.4378	0.4129
			SSE	0.4415	0.4324	0.3981	0.3444
			SMSE	0.2122	0.1994	0.1623	0.1262
		σ_η^2	SM	0.2401	0.2686	0.2472	0.2533
			SSE	0.4496	0.3928	0.2620	0.2204
			SMSE	0.2023	0.1546	0.0687	0.0486
0.5	0.25	γ_1	SM	0.3105	0.2683	0.2619	0.2336
			SSE	0.4388	0.4078	0.3629	0.3226
			SMSE	0.1962	0.1667	0.1318	0.1043
		σ_η^2	SM	0.4505	0.4815	0.4696	0.4746
			SSE	0.6181	0.5632	0.4679	0.3792
			SMSE	0.3845	0.3175	0.2199	0.1444

(Table 3.2 Contd....)

σ_η^2	γ_1	Parameters	Quantity	Time Series Length (T)			
				100	200	300	500
0.5	0.5	γ_1	SM	0.3568	0.3657	0.4101	0.4064
			SSE	0.4139	0.3801	0.3341	0.2916
			SMSE	0.1918	0.1625	0.1197	0.0938
		σ_η^2	SM	0.4878	0.5408	0.5164	0.5230
			SSE	0.5487	0.5329	0.4066	0.3295
			SMSE	0.3013	0.2857	0.1656	0.1091
1.0	0.25	γ_1	SM	0.2274	0.1855	0.1996	0.1945
			SSE	0.4047	0.3439	0.3051	0.2588
			SMSE	0.1643	0.1224	0.0956	0.0701
		σ_η^2	SM	0.9216	0.9528	0.9352	0.9705
			SSE	0.8317	0.6446	0.4838	0.4186
			SMSE	0.6979	0.4177	0.2382	0.1761
1.0	0.5	γ_1	SM	0.3352	0.3315	0.3874	0.4001
			SSE	0.3808	0.3401	0.3152	0.2776
			SMSE	0.1722	0.1441	0.1120	0.0870
		σ_η^2	SM	1.0169	1.0630	1.0504	1.0451
			SSE	0.8264	0.6980	0.6321	0.5239
			SMSE	0.6832	0.4912	0.4021	0.2765

3.2 Large Sample Case

In the large sample case, the SMM does not depend on the initial variance σ_1^2 . However, the WGQL approach still depends on σ_1^2 . We use the same value $\sigma_1^2 = \sigma_{i0}^2 = 1.25$ as in the small sample case.

For the SMM estimation of γ_1 and σ_η^2 , we solve the asymptotic estimating equations (2.22) and (2.13) for γ_1 and σ_η^2 , respectively. As far as the large sample estimation by the WGQL approach is concerned, it is clear that the WGQL estimating equations (2.44) for γ_1 , and (2.59) for σ_η^2 , used in the small sample case, is still valid for the large sample. The large sample based performances of the SMM and WGQL approaches are reported in Table 3.3 and 3.4, respectively.

Table 3.3: Simulated mean (SM), simulated standard error (SSE) and simulated mean square error (SMSE) of the SMM estimates based on large time series for selected parameters values by using 1000 simulations.

Parameters	Quantity	Time Series Length (T)				
		1000	2000	3000	6000	10,000
$\gamma_1 = 0.25$	SM	0.2476	0.2385	0.2435	0.2302	0.2442
	SSE	0.3319	0.2473	0.2018	0.1428	0.1174
	SMSE	0.1102	0.0613	0.0408	0.0208	0.0138
$\sigma_\eta^2 = 0.25$	SM	0.2265	0.2396	0.2396	0.2476	0.2471
	SSE	0.1203	0.0874	0.0723	0.0486	0.0402
	SMSE	0.0150	0.0078	0.0053	0.0024	0.0016
$\gamma_1 = 0.5$	SM	0.4213	0.4462	0.4537	0.4475	0.4615
	SSE	0.2832	0.1969	0.1628	0.1171	0.0965
	SMSE	0.0864	0.0417	0.0286	0.0165	0.0108
$\sigma_\eta^2 = 0.25$	SM	0.2540	0.2596	0.2572	0.2635	0.2600
	SSE	0.1381	0.1021	0.0856	0.0595	0.0498
	SMSE	0.0191	0.0105	0.0074	0.0037	0.0026
$\gamma_1 = 0.25$	SM	0.2067	0.2283	0.2386	0.2323	0.2451
	SSE	0.2309	0.1690	0.1402	0.0999	0.0836
	SMSE	0.0552	0.0290	0.0198	0.0103	0.0070
$\sigma_\eta^2 = 0.5$	SM	0.4768	0.4913	0.4885	0.4989	0.4966
	SSE	0.1505	0.1071	0.0877	0.0593	0.0489
	SMSE	0.0232	0.0115	0.0078	0.0035	0.0024

(Table 3.3 Contd....)

Parameters	Quantity	Time Series Length (T)				
		1000	2000	3000	6000	10,000
$\gamma_1 = 0.5$	SM	0.4205	0.4441	0.4537	0.4482	0.4623
	SSE	0.2191	0.1659	0.1390	0.0964	0.0811
	SMSE	0.0543	0.0307	0.0215	0.0120	0.0080
$\sigma_\eta^2 = 0.5$	SM	0.5132	0.5181	0.5136	0.5264	0.5192
	SSE	0.2024	0.1541	0.1283	0.0854	0.0723
	SMSE	0.0412	0.0241	0.0167	0.0080	0.0056
$\gamma_1 = 0.25$	SM	0.1948	0.2199	0.2334	0.2308	0.2442
	SSE	0.1960	0.1508	0.1305	0.0950	0.0798
	SMSE	0.0415	0.0236	0.0173	0.0094	0.0064
$\sigma_\eta^2 = 1.0$	SM	0.9730	0.9905	0.9815	0.9979	0.9933
	SSE	0.2257	0.1625	0.1361	0.0941	0.0760
	SMSE	0.0517	0.0265	0.0189	0.0089	0.0058
$\gamma_1 = 0.5$	SM	0.4109	0.4433	0.4577	0.4541	0.4738
	SSE	0.2135	0.1909	0.1738	0.1374	0.1320
	SMSE	0.0535	0.0397	0.0320	0.0210	0.0181
$\sigma_\eta^2 = 1.0$	SM	1.0328	1.0161	1.0021	1.0292	1.0069
	SSE	0.3636	0.3220	0.2923	0.2305	0.2261
	SMSE	0.1333	0.1039	0.0854	0.0540	0.0512

Table 3.4: Simulated mean (SM), simulated standard error (SSE) and simulated mean square error (SMSE) of WGQL estimates based on large time series for selected parameters values by using 1000 simulations.

Parameters	Quantity	Time Series Length (T)		
		1000	2000	3000
$\gamma_1 = 0.25$	SM	0.2574	0.2413	0.2422
	SSE	0.3282	0.2502	0.2038
	SMSE	0.1078	0.0627	0.0416
$\sigma_v^2 = 0.25$	SM	0.2284	0.2400	0.2402
	SSE	0.1308	0.0882	0.0722
	SMSE	0.0176	0.0079	0.0053
$\gamma_1 = 0.5$	SM	0.4280	0.4467	0.4564
	SSE	0.2873	0.2077	0.1698
	SMSE	0.0877	0.0460	0.0372
$\sigma_v^2 = 0.25$	SM	0.2567	0.2607	0.2570
	SSE	0.1557	0.1053	0.0887
	SMSE	0.0243	0.0112	0.0079
$\gamma_1 = 0.25$	SM	0.2111	0.2293	0.2384
	SSE	0.2325	0.1720	0.1408
	SMSE	0.0556	0.0300	0.0199
$\sigma_v^2 = 0.5$	SM	0.4804	0.4914	0.4894
	SSE	0.1606	0.1092	0.0870
	SMSE	0.0262	0.0120	0.0076

(Table 3.4 Contd....)

Parameters	Quantity	Time Series Length (T)		
		1000	2000	3000
$\gamma_1 = 0.5$	SM	0.4251	0.4474	0.4549
	SSE	0.2256	0.1739	0.1431
	SMSE	0.0565	0.0330	0.0251
$\sigma_v^2 = 0.5$	SM	0.5289	0.5262	0.5160
	SSE	0.2931	0.2219	0.1377
	SMSE	0.0287	0.0499	0.0192
$\gamma_1 = 0.25$	SM	0.1962	0.2211	0.2347
	SSE	0.1975	0.1563	0.1365
	SMSE	0.0419	0.0253	0.0189
$\sigma_v^2 = 1.0$	SM	0.9788	1.0027	0.9883
	SSE	0.2278	0.2832	0.2390
	SMSE	0.0524	0.0802	0.0573
$\gamma_1 = 0.5$	SM	0.4326	0.4587	0.4775
	SSE	0.2491	0.2188	0.2056
$\sigma_v^2 = 1.0$	SM	1.0753	1.0797	1.0484
	SSE	0.6351	0.5972	0.5676
	SMSE	0.4090	0.3630	0.3249

3.3 Interpretation of the Small and Large Sample Simulation Results

As far as the small sample performance is concerned, both SMM and WGQL approaches provides some what reasonable, but not so satisfactory estimates. For example, when $T = 500$, the SMM approach provides estimates for $\gamma_1 = 0.5$ and $\sigma_\eta^2 = 0.5$ as $\hat{\gamma}_{1,SMM} = 0.40$ and $\hat{\sigma}_{\eta,SMM}^2 = 0.51$, respectively, with corresponding simulated standard error 0.28 and 0.25. For the same parameter values, the WGQL provides $\hat{\gamma}_{1,WGQL} = 0.41$ with its simulated standard error 0.29 and $\hat{\sigma}_{\eta,WGQL}^2 = 0.52$ with its standard error 0.33. These and other results in Tables 3.1 and 3.2 indicate that the estimates of σ_η^2 appears to be close to the true values whereas the estimates of γ_1 are not so satisfactory. But, the estimates of γ_1 get closer to the true values when the length of the series is increased.

The Table 3.3 show that for a reasonably large time series with length between $T = 1000$ and $10,000$, the proposed SMM approach performs very well in estimating both γ_1 and σ_η^2 parameters. This is a big improvement over the existing GMM and QML approaches mainly because of the fact that proposed SMM approach is simpler and computationally quite efficient. Also, unlike the existing GMM and QML approaches, the SMM approach does not encounter any convergence problems. To be specific, when $T = 3000$, for example, the SMM approach provides estimates for

$\gamma_1 = 0.5$ and $\sigma_\eta^2 = 1.0$ as $\hat{\gamma}_{1,SMM} = 0.46$ with its simulated standard error 0.17 and $\hat{\sigma}_{\eta,SMM}^2 = 1.00$ with its standard error 0.29. For the same parameter values, when $T = 10,000$, the SMM approach produces $\hat{\gamma}_{1,SMM} = 0.47$ with its simulated standard error 0.13 and $\hat{\sigma}_{\eta,SMM}^2 = 1.00$ with its standard error 0.23. Thus, it is clear that the SMM approach works very well even if the length of the series is small as $T = 3000$. However, as expected, the standard errors or mean squared errors of the estimates improves substantially when T increased from 3000 to 10,000.

The results in Table 3.4 show that the proposed WGQL approach performs similarly to the SMM approach. Note however, that to save time and space we have considered $T = 1000, 2000$ and 3000, in this case. As the length of the series increases, both SMM and WGQL approach appears to perform better as expected. As mentioned earlier, the WGQL approach behaves similarly to the SMM approach. For example, for the same parameter values, when $T = 3000$ the WGQL estimates for $\hat{\gamma}_{1,WGQL} = 0.48$ with its simulated standard error 0.21 and $\hat{\sigma}_{\eta,WGQL}^2 = 1.05$ with its simulated standard error 0.57. Thus WGQL approach appears to produce same or better estimates for γ_1 and σ_η^2 , but with relatively larger standard errors. For this and similar other reasons, between the proposed SMM and WGQL approach, we prefer the SMM approach over the WGQL approach.

Note that the asymptotic variances for the estimators of γ_1 and σ_η^2 reported in Table 2.1 in Chapter 2 are in agreement with the corresponding simulated variances

reported in Table 3.3 and 3.4 for the SMM and WGQL approaches. Thus, when it is required, one may estimate the standard errors of the estimates by using the formulas for the asymptotic standard deviations.

3.4 True Versus Estimated Kurtosis under the SV Model

To understand the volatility, that is, to realize the changes in variance pattern in the time series, it is recommended to examine the kurtosis of the data over time. See, for example, Jacquier et.al (1994, p.387) Shephard (1996, p.23), Mills (1999, p.129), Ruiz (2004, p.615) and Tsay (2005, p.134)). For the purpose, in Lemma (3.4.1) below, we provide a general formula for the kurtosis under the volatility model (1.1)-(1.2).

Lemma 3.4.1. *Kurtosis for $\{y_t\}$ under the volatility model (1.1)-(1.2) is given by*

$$\kappa_t(\gamma_1, \sigma_\eta^2) = \begin{cases} 3 & \text{for } t = 1 \\ \frac{3 \left[\exp \left(2 \gamma_1^{t-1} \log \sigma_1^2 + 2 \sigma_\eta^2 \sum_{r=0}^{t-2} \gamma_1^{2r} \right) \right]}{\left[\exp \left(\gamma_1^{t-1} \log \sigma_1^2 + \frac{\sigma_\eta^2}{2} \sum_{r=0}^{t-2} \gamma_1^{2r} \right) \right]^2} & \text{for } t = 2, \dots, T, \end{cases} \quad (3.1)$$

To prove the lemma, we first compute $E[Y_t^4]$ by (2.32) and $E[Y_t^2]$ by (2.29).

The results in the lemma are immediate from the formula for the kurtosis given by

$$\kappa_t(\gamma_1, \sigma_\eta^2) = \frac{3E[Y_t^4]}{[E(Y_t^2)]^2}.$$

Note that, in the limiting case, i.e, when $t \rightarrow \infty$, the kurtosis in (3.1) reduces to

$$\lim_{t \rightarrow \infty} \kappa_t(\gamma_1, \sigma_\eta^2) = 3 \exp \left\{ \frac{\sigma_\eta^2}{1 - \gamma_1^2} \right\}, \quad (3.2)$$

which agrees with the formula for kurtosis studied by Harvey et.al (1994, p.249), Mills (1999, p.249) and Broto and Ruiz (2004, p.615), among others. Further note that, the formula for the kurtosis given in (3.2) is independent of time and it is a function of the volatility parameters γ_1 and σ_η^2 , whereas kurtosis at a finite time point given by (3.1) is dependent on first few times and it is a function of γ_1 , σ_η^2 and σ_1^2 . Now to understand the effects of the parameters γ_1 , σ_η^2 and σ_1^2 on the kurtosis, we, for example, display the true kurtosis computed by (3.1) in Figures 3.1 and 3.2 for selected values of the parameters. In the same figures we also display the estimated kurtosis computed by using $\hat{\gamma}_{1, SMM}$ and $\hat{\sigma}_{\eta, SMM}^2$ for γ_1 and σ_η^2 respectively. As far as the initial variance σ_1^2 is concerned, we have chosen $\sigma_1^2 = 1.25$.

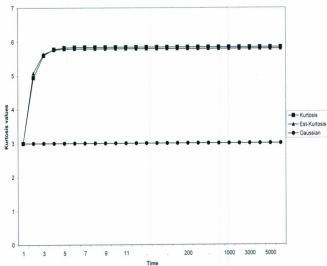


Figure 3.1: True and estimated kurtosis with volatility parameters $\gamma_1 = 0.5$, $\sigma_\eta^2 = 0.5$.

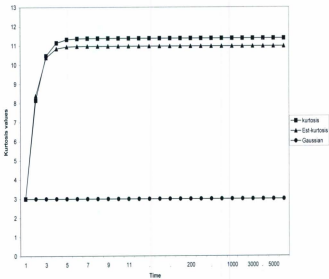


Figure 3.2: True and estimated kurtosis with volatility parameters $\gamma_1 = 0.5$, $\sigma_\eta^2 = 1.0$.

It is clear from Figures 3.1 and 3.2 that the kurtosis under the present volatility model (1.1)-(1.2) is much larger than the Gaussian based kurtosis (=3). These figures also exhibit that the kurtosis gets stabilized quickly after an initial short period. To be specific, Figure 3.1 shows that when $\gamma_1 = 0.5$ and $\sigma_\eta^2 = 0.5$, the kurtosis gets stabilized at $\kappa_t = 5.8432$ for any $t > 4$. Similarly, Figure 3.2 shows that when $\gamma_1 = 0.5$ and $\sigma_\eta^2 = 1.0$, the kurtosis gets stabilized at $\kappa_t = 11.3810$ for any $t > 4$.

Note that, since in practice, true kurtosis is unknown, as mentioned above, we have also estimated the kurtosis by using the estimates of the parameters in the formula for kurtosis given in (3.1), and the estimated kurtosis are displayed in Figures 3.1 and 3.2. The estimated kurtosis appears to be very close to the the corresponding true of the kurtosis, indicating that the proposed SMM technique performs very well in estimating the parameters of the volatility model.

Chapter 4

Extended Stochastic Volatility

Models

4.1 Model and the Properties

In Chapters from 1 to 3, we have discussed the inferences in the stochastic volatility (SV) model. Recall that under the SV model (1.1)-(1.2), the responses $\{y_t\}$ are uncorrelated. That is, for $u < t$

$$\begin{aligned} E[Y_u Y_t] &= E_{\sigma_1, \dots, \sigma_t} E[\sigma_u \sigma_t \epsilon_u \epsilon_t | \sigma_1, \dots, \sigma_t] \\ &= E_{\sigma_1, \dots, \sigma_t} [\sigma_u \sigma_t E[\epsilon_u \epsilon_t | \sigma_1, \dots, \sigma_t]] \\ &= E_{\sigma_1, \dots, \sigma_t} [\sigma_u \sigma_t E[\epsilon_u \epsilon_t]] \\ &= 0, \end{aligned} \tag{4.1}$$

as ϵ_t and σ_t are independent for all t and also $\epsilon_t \stackrel{iid}{\sim} N(0, 1)$. Consequently,

$$\text{Cov}(Y_u, Y_t) = E[Y_u Y_t] - E[Y_u]E[Y_t] = 0. \quad (4.2)$$

But in practice, it may happen that, conditional on the variances, the time series observations may be correlated. This type of data can be modeled by using the relationship

$$y_t = x_t' \beta + \theta(y_{t-1} - x_{t-1}' \beta) + \sigma_t \epsilon_t, \quad t = 2, \dots, T, \quad (4.3)$$

with $y_t = x_t' \beta + \sigma_t \epsilon_t$, where $x_t = (x_{t1}, \dots, x_{tp})'$ is a p -dimensional (say) vector of time dependent covariates and β is the corresponding regression effect. In (4.3), θ is a scalar dynamic dependence parameter. Also under in (4.3), σ_t^2 follows the original volatility model as in (1.2), i.e.,

$$\ln(\sigma_t^2) = h_t = \gamma_0 + \gamma_1 h_{t-1} + \eta_t; \quad t = 2, \dots, T. \quad (4.4)$$

Thus, the model (4.3) accommodates the dynamic relationship of the responses. Furthermore, this model are modeled through another dynamic relationship as given in (4.4) (see also (1.2)). Consequently, we refer to the complete model (4.3)-(4.4) as an observation driven dynamic - dynamic (ODDD) stochastic volatility model.

The new ODDD model (4.3) - (4.4), conditional on the variances yields the conditional mean, variance and pairwise covariances as in the following lemma. As before we consider $\gamma_0 = 0$ for simplicity.

Lemma 4.1.1. When $\{y_t, t = 1, \dots, T\}$ follows the model (4.3), one obtains the expectation and variance of y_t conditional on $\sigma_1, \dots, \sigma_t$ as

$$E(Y_t | \sigma_1, \dots, \sigma_t) = x_t' \beta, \quad (4.5)$$

and

$$\text{Var}(Y_t | \sigma_1, \dots, \sigma_t) = \sum_{j=1}^t \theta^{2(t-j)} \sigma_j^2, \quad (4.6)$$

respectively. Furthermore, for $u < t$ the conditional covariance between y_u and y_t is given by

$$\text{Cov}(Y_u, Y_t | \sigma_1, \dots, \sigma_t) = \sum_{j=1}^u \theta^{t+u-2j} \sigma_j^2 \quad (4.7)$$

Proof. The proof is simple. Nevertheless, it is shown in Appendix C. \square

Note that when $\theta=0$, the conditional variance (4.6) of y_t reduces to

$$\text{Var}(Y_t | \sigma_t) = \sigma_t^2, \quad (4.8)$$

and the conditional covariance between y_u and y_t (4.7) becomes zero. That is,

$$\text{Cov}(Y_u, Y_t | \sigma_1, \dots, \sigma_t) = 0. \quad (4.9)$$

As expected, these conditional variance and covariance also follow directly from the SV model (1.1)-(1.2). In all other cases i.e. when $\theta \neq 0$, it is clear that unlike the SV model, the observations are correlated both conditionally and unconditionally the conditional covariances being given by (4.7). The unconditional covariances are given in Lemma 4.1.2 along with unconditional means and variances.

Lemma 4.1.2. For all $t = 1, \dots, T$, the response y_t has the unconditional mean and variance given by

$$E(Y_t) = x_t' \beta = \mu_t \quad (\text{say}), \quad (4.10)$$

$$\begin{aligned} \text{Var}[Y_t] &= \sum_{j=1}^t \theta^{2(t-j)} E_M[\sigma_j^2], \\ &= \varphi_{tt} \quad (\text{say}). \end{aligned} \quad (4.11)$$

For $u < t$, the unconditional covariance between y_u and y_t is given by

$$\begin{aligned} \text{Cov}[Y_u, Y_t] &= \sum_{j=1}^u \theta^{t+u-2j} E_M[\sigma_j^2], \\ &= \varphi_{ut} \quad (\text{say}), \end{aligned} \quad (4.12)$$

where $E_M[\sigma_j^2]$ for a given time j is already given in (2.8) following the dynamic relationship (1.2) [see also (4.4)]. That is,

$$E_M(\sigma_j^2) = \exp\left(\gamma_1^{j-1} \ln \sigma_{10}^2 + \frac{1}{2} \sigma_\eta^2 \sum_{r=0}^{j-2} \gamma_1^{2r}\right) \quad j = 2, \dots, t \quad t = 2, \dots, T. \quad (4.13)$$

Proof. The unconditional expectation in (4.10) follows from the fact that, the conditional mean in (4.5) is free from $\sigma_1^2, \dots, \sigma_t^2, \dots, \sigma_T^2$.

The unconditional covariance is obtained by using the formula

$$\begin{aligned} \text{Cov}[Y_u, Y_t] &= E_{\sigma_1, \dots, \sigma_t} [\text{Cov}(Y_u, Y_t) | \sigma_1, \dots, \sigma_t] \\ &\quad + \text{Cov}_{\sigma_1, \dots, \sigma_t} [E(Y_u | \sigma_1, \dots, \sigma_t), E(Y_t | \sigma_1, \dots, \sigma_t)]. \end{aligned} \quad (4.14)$$

Since the conditional expectations shown in (4.5) are free from σ_t ($t = 1, \dots, T$), the second term in (4.14) is zero. Furthermore, using the formula for the conditional covariance from (4.7), the first term in (4.14) may be evaluated by computing $E(\sigma_j^2)$ for $j = 2, \dots, u$. Thus, we write the unconditional covariance between Y_u and Y_t as

$$\text{Cov}[Y_u, Y_t] = \sum_{j=1}^u \theta^{t+u-2j} E_M[\sigma_j^2], \quad (4.15)$$

yielding the variance for $u = t$ as

$$\text{Var}(Y_t) = \sum_{j=1}^t \theta^{2(t-j)} E_M[\sigma_j^2], \quad (4.16)$$

where $E_M[\sigma_j^2]$ is as in (4.13). \square

4.2 A Remarks on Stationarity

Note that, when $|\theta| < 1$ and $|\gamma_1| < 1$ in the model (4.3)-(4.4), limiting variances as $t \rightarrow \infty$ reduces to a finite constant. This is because,

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Var}(Y_t) &= \lim_{t \rightarrow \infty} \sum_{j=1}^t \theta^{2(t-j)} E_M[\sigma_j^2] \\ &= \lim_{t \rightarrow \infty} E_M[\sigma_t^2] \lim_{t \rightarrow \infty} \sum_{j=1}^t \theta^{2(t-j)}, \end{aligned} \quad (4.17)$$

Since $\lim_{t \rightarrow \infty} E[\sigma_t^2] = \exp\left[\frac{\sigma_\eta^2}{2} \left(\frac{1}{1-\gamma_1^2}\right)\right]$, by (2.9) we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Var}(Y_t) &= \exp\left[\frac{\sigma_\eta^2}{2} \left(\frac{1}{1-\gamma_1^2}\right)\right] \left[\lim_{t \rightarrow \infty} \left\{ \theta^{2(t-1)} + \theta^{2(t-2)} + \theta^{2(t-3)} + \dots + \theta^6 + \theta^4 + \theta^2 + 1 \right\} \right] \\ &= \exp\left[\frac{\sigma_\eta^2}{2} \left(\frac{1}{1-\gamma_1^2}\right)\right] \left[\lim_{t \rightarrow \infty} \left\{ 1 + \theta^2 + \theta^4 + \theta^6 + \dots + \theta^{2(t-1)} \right\} \right] \\ &= \exp\left[\frac{\sigma_\eta^2}{2} \left(\frac{1}{1-\gamma_1^2}\right)\right] \left[\frac{1}{1-\theta^2} \right]. \end{aligned} \quad (4.18)$$

Note that this limiting variances may be useful in developing asymptotic estimation. Furthermore, when $\theta = 0$, the limiting variance in (4.18) reduces to the stationary variances under the SV model.

4.3 Estimation of the Parameters in ODDD Model

4.3.1 GQL approach

The model (4.3)- (4.4) involves (i) β , the p - dimensional vector of regression parameters (ii) θ , the dynamic dependence parameter (iii) γ_1 , dynamic volatility parameter and (iv) σ_v^2 volatility variance parameter. In Chapter 2, we have estimated γ_1 and σ_v^2 under the SV model. In the present model β and θ are additional and important parameters. More specifically, β is involved in the means of the responses, and θ, γ_1 and σ_v^2 are involved in the variances and covariances of the responses.

Note that, β is clearly a vector of regression parameters. As far as θ is concerned, conditional on the past responses, it may also be treated as a regression parameter. For this reason, we estimate both β and θ by using a GQL (Sutradhar (2004)) approach originally developed for the estimation of the parameters in mixed model set up. The other paramters namely, γ_1 and σ_v^2 will be estimated by using the SMM approach that we exploited for the inferences in the SV model.

4.3.2 GQL estimating equation for β

Note that the unconditional mean μ_t for Y_t in (4.10) is a function of β , whereas for $u \leq t$, the unconditional second order moments, namely $Cov(Y_u, Y_t)$ in (4.12) are the functions of the other parameters θ, γ_1 , and σ_η^2 . Thus, we construct a basic statistic using $y = (y_1, y_2, \dots, y_t, \dots, y_T)'$ to estimate β involved in $\mu = E[Y] = (\mu_1, \mu_2, \dots, \mu_t, \dots, \mu_T)'$, where by (4.1)-(4.2) $\mu_t = x_t'\beta$, with $x_t = (x_{t1}, \dots, x_{tj}, \dots, x_{tp})'$. Let $\Sigma = (\varphi_{ut})$ be the covariance matrix of y with $\varphi_{ut} = Cov(Y_u, Y_t)$ as in (4.12). Now, for given $\xi = (\theta, \alpha')' = (\theta, \gamma_1, \sigma_\eta^2)'$, i.e. for given Σ , by following Sutradhar (2004), one may easily obtain a GQL estimate of β by solving the estimating equation

$$\frac{\partial \mu'}{\partial \beta} \Sigma^{-1} (y - \mu) = 0. \quad (4.19)$$

Let $\hat{\beta}_{GQL}$ be the solution of (4.19), and the iterative equation for $\hat{\beta}_{GQL}$ is given by

$$\hat{\beta}_{GQL}(r+1) = \hat{\beta}_{GQL}(r) + \left[\left(\frac{\partial \mu'}{\partial \beta} \Sigma^{-1} \frac{\partial \mu}{\partial \beta} \right)^{-1} \left(\frac{\partial \mu'}{\partial \beta} \Sigma^{-1} (y - \mu) \right) \right]_{\hat{\beta}_{GQL}(r)}, \quad (4.20)$$

where to compute the first order derivative $\frac{\partial \mu'}{\partial \beta}$ in (4.20) it is sufficient to compute the derivative vector $\frac{\partial \mu_t}{\partial \beta}$ for all $t=1, \dots, T$. This derivative vector has the expression $\frac{\partial \mu_t}{\partial \beta} = x_t$, and $\hat{\beta}_{GQL}(r)$ denotes the GQL estimate of β as a solution of (4.20) at the r -th iteration, and $[\cdot]_{\hat{\beta}_{GQL}(r)}$ is the value of the expression in the square bracket evaluated at $\beta = \hat{\beta}_{GQL}(r)$.

4.3.3 GQL estimating equation for θ

Since θ may be treated as a regression parameter conditional on the past lag1 responses, we construct the basic statistic using a vector of lag 1 based corrected pairwise products of the responses. Let

$$s_2 = [(y_1 - \mu_1)(y_2 - \mu_2), \dots, (y_{t-1} - \mu_{t-1})(y_t - \mu_t), \dots, (y_{T-1} - \mu_{T-1})(y_T - \mu_T)]', \quad (4.21)$$

and $\lambda_2^* = E[s_2]$ and $\Delta = \text{Cov}(s_2)$. For given θ, γ_1 and σ_0^2 , we first estimate β parameter by using (4.19) - (4.20). Once we get the estimate of β , we estimate θ , by solving the GQL estimating equation

$$\frac{\partial \lambda_2^*}{\partial \theta} \Delta^{-1} (s_2 - \lambda_2^*) = 0. \quad (4.22)$$

As far as the formulas for λ_2^* is concerned, we write

$$\begin{aligned} \lambda_2^* &= [E[(y_1 - \mu_1)(y_2 - \mu_2)], \dots, E[(y_{t-1} - \mu_{t-1})(y_t - \mu_t)], \dots, \\ &\quad E[(y_{T-1} - \mu_{T-1})(y_T - \mu_T)]]' \\ &= [\varphi_{12}, \varphi_{23}, \dots, \varphi_{t-1,t}, \dots, \varphi_{T-1,T}]', \end{aligned}$$

with $\varphi_{t-1,t}$ as in (4.12).

The derivation of the formulas for the elements of Δ is complicated. Nevertheless,

we provide the formulas for the diagonal elements of Δ as follows. Also an outline is given for the computation of the off diagonal elements.

Computation of the diagonal elements of Δ .

Note that conditional on $\sigma_1^2, \dots, \sigma_T^2$, it is clear from (4.3) that the responses follows T- dimensional normal distribution. Thus, conditional on variances one obtains

$$E \left[(Y_i - \mu_i)(Y_j - \mu_j)(Y_k - \mu_k)(Y_l - \mu_l) | \sigma_1^2, \dots, \sigma_i^2, \dots, \sigma_T^2 \right] = \varphi_{ij}^* \varphi_{kl}^* + \varphi_{ik}^* \varphi_{jl}^* + \varphi_{il}^* \varphi_{jk}^*, \quad (4.23)$$

where $\varphi_{ot}^* = \sum_{j=1}^n \theta^{t+u-2j} \sigma_j^2$. For the computation of the diagonal elements of Δ we write

$$\begin{aligned} \text{Var}[(Y_{t-1} - \mu_{t-1})(Y_t - \mu_t)] &= E[(Y_{t-1} - \mu_{t-1})^2(Y_t - \mu_t)^2] \\ &\quad - (E[(Y_{t-1} - \mu_{t-1})(Y_t - \mu_t)])^2 \\ &= E_{\sigma_1^2, \dots, \sigma_t^2} \left[\varphi_{t-1,t}^* \varphi_{t-1,t}^* + \varphi_{t-1,t-1}^* \varphi_{t,t}^* + \varphi_{t-1,t}^* \varphi_{t,t-1}^* \right] \\ &\quad - \varphi_{t-1,t}^2, \end{aligned} \quad (4.24)$$

where $E_{\sigma_1^2, \dots, \sigma_t^2} [\varphi_{t-1,t}^* \varphi_{t-1,t}^* + \varphi_{t-1,t-1}^* \varphi_{t,t}^* + \varphi_{t-1,t}^* \varphi_{t,t-1}^*]$ given in Appendix C.

An outline for off diagonal elements of Δ

For $u < t$, the off diagonal elements of Δ has the formula as

$$\begin{aligned} \text{Cov}[(Y_{u-1} - \mu_{u-1})(Y_u - \mu_u), (Y_{t-1} - \mu_{t-1})(Y_t - \mu_t)] \\ = E[(Y_{u-1} - \mu_{u-1})(Y_u - \mu_u)(Y_{t-1} - \mu_{t-1})(Y_t - \mu_t)] \end{aligned}$$

$$\begin{aligned}
& -E\left[(Y_{u-1} - \mu_{u-1})(Y_u - \mu_u)\right]E\left[(Y_{t-1} - \mu_{t-1})(Y_t - \mu_t)\right] \\
& = E_{\sigma_1^2, \dots, \sigma_t^2} \left[\varphi_{u-1,u}^* \varphi_{t-1,t}^* + \varphi_{u-1,t-1}^* \varphi_{u,t}^* + \varphi_{u-1,t}^* \varphi_{u,t-1}^* \right] - \varphi_{u-1,u} \varphi_{t-1,t}, \quad (4.25)
\end{aligned}$$

where $E_{\sigma_1^2, \dots, \sigma_t^2} \left[\varphi_{u-1,u}^* \varphi_{t-1,t}^* + \varphi_{u-1,t-1}^* \varphi_{u,t}^* + \varphi_{u-1,t}^* \varphi_{u,t-1}^* \right]$ given in Appendix C.

Further note that the construction of the GQL estimating equation (4.22) also requires the formula for the derivative of λ_2^* with respect to the elements of θ , that is,

$$\frac{\partial \varphi_{ut}'}{\partial \theta} = \sum_{j=1}^u (t+u-2j) \theta^{t+u-2j-1} E_M[\sigma_j^2]. \quad (4.26)$$

Let $\hat{\theta}_{GQL}$ be the GQL estimator of θ obtained from (4.22). Similar to (4.20), $\hat{\theta}_{GQL}$ is obtained by using the iterative equation

$$\hat{\theta}_{GQL}(r+1) = \hat{\theta}_{GQL}(r) + \left[\left(\frac{\partial \lambda_2^*}{\partial \theta} \Delta^{-1} \frac{\partial \lambda_2^*}{\partial \beta} \right)^{-1} \left(\frac{\partial \lambda_2^*}{\partial \theta} \Delta^{-1} (s_2 - \lambda_2^*) \right) \right]_{\hat{\theta}_{GQL}(r)}, \quad (4.27)$$

where $\hat{\theta}_{GQL}(r)$ denotes the GQL estimate of θ as a solution of (4.27) at the r -th iteration, and $[\cdot]_{\hat{\theta}_{GQL}(r)}$ is the value of the expression in the square bracket evaluated at $\theta = \hat{\theta}_{GQL}(r)$.

4.3.4 SMM estimating equation for $\alpha = (\gamma_1, \sigma_\eta^2)'$

(a) Unbiased Estimating Equation for γ_1

Here, the estimation of β, θ and α will be done in cycles of iterations. For given θ and α , we first estimate β by using (4.19) - (4.20). Once we get this estimate, we use the

GQL iterative estimating equation (4.27) for θ . For moment estimation of γ_1 , similar to S_2 in Section 2.1.1.2, under the SV model, we consider a moment function

$$S_3 = \frac{1}{T-1} \sum_{t=2}^T (y_{t-1} - x'_{t-1}\beta)^2 (y_t - x'_t\beta)^2, \quad (4.28)$$

and solve

$$S_3 - E[S_3] = 0, \quad (4.29)$$

for given β, θ and σ_η^2 . Here $E[S_3] = \frac{1}{T-1} \sum_{t=2}^T \bar{\varphi}_{t-1,t}$, with $\bar{\varphi}_{t-1,t} = E[(y_{t-1} - x'_{t-1}\beta)^2 (y_t - x'_t\beta)^2]$. The formula for $\bar{\varphi}_{t-1,t}$ given in Appendix C.

By using Taylor's series expansion, it follows from (4.29) that the γ_1 parameter may be estimated by using the iterative equation

$$\hat{\gamma}_{1,MM}(r+1) = \hat{\gamma}_{1,MM}(r) + \left[\left(\frac{\partial E[S_3]}{\partial \gamma_1} \right)^{-1} (S_3 - E[S_3]) \right]_{\hat{\gamma}_{1,MM}(r)}, \quad (4.30)$$

where $\hat{\gamma}_{1,MM}(r)$ denotes the moment estimate of γ_1 as a solution of (4.30) at the r -th iteration, and $[\cdot]_{\hat{\gamma}_{1,MM}(r)}$ is the value of the expression in the square bracket evaluated at $\gamma_1 = \hat{\gamma}_1(r)$. Note that the equation (4.30) requires the expression of $E[S_3]$ and computation of the derivative $\frac{\partial E[S_3]}{\partial \gamma_1}$, which are given in Appendix C.

(b) Unbiased Estimating Equation for σ_η^2

For the moment estimation of σ_η^2 under the ODDD volatility model, we consider the moment function

$$S_4 = \frac{1}{T} \sum_{t=1}^T (y_t - x'_t\beta)^2, \quad (4.31)$$

which is similar to S_1 in Section 2.1.1.1, under the SV model. Now, we use the improved estimates of β, θ and γ_1 and we solve the moment estimating equation

$$S_4 - E[S_4] = 0 \quad (4.32)$$

for σ_η^2 , where $E[(y_t - x_t' \beta)^2]$ has the expression $\left(\sum_{j=1}^t \theta^{2(t-j)} E_M[\sigma_j^2] \right)$ [see also (4.16)] with

$$E_M[\sigma_j^2] = \exp\left(\gamma_1^{j-1} \log \sigma_1^2 + \frac{1}{2} \sigma_\eta^2 \sum_{r=0}^{j-2} \gamma_1^{2r}\right) \quad j = 2, \dots, t \quad t = 2, \dots, T.$$

The estimating equation (4.32) may be solved iteratively by using

$$\hat{\sigma}_\eta^2[r+1] = \hat{\sigma}_\eta^2[r] + \left[\left(\frac{\partial E[S_4]}{\partial \sigma_\eta^2} \right)^{-1} (S_4 - E[S_4]) \right]_{\hat{\sigma}_{\eta, MM}^2(r)}, \quad (4.33)$$

where $\hat{\sigma}_{\eta, MM}^2(r)$ denotes the moment estimate of σ_η^2 as a solution of (4.33) at the r -th iteration, and $[\cdot]_{\hat{\sigma}_{\eta, MM}^2(r)}$ is the value of the expression in the square bracket evaluated at $\sigma_\eta^2 = \hat{\sigma}_\eta^2(r)$. Note that the equation (4.33) requires the computation of derivative $\frac{\partial E[S_4]}{\partial \sigma_\eta^2}$, which has the formula given by

$$\frac{\partial E[S_4]}{\partial \sigma_\eta^2} = \left[\sum_{j=1}^t \theta^{2(t-j)} E_M[\sigma_j^2] \left\{ \frac{1}{2} \sum_{r=0}^{j-2} \gamma_1^{2r} \right\} \right]$$

We now summarize the aforementioned estimation steps for all parameters and give the following algorithm.

Algorithm for ODDD Volatility Model:

Step 1: For initial values of $\theta, \gamma_1, \sigma_\eta^2$, and β first estimate of β is obtained from (4.20).

Step 2: The improved estimate of β obtained from step 1 is used in (4.27) along with

initial values of γ_1 and σ_η^2 to obtain an improved estimate of θ .

Step 3: The improved estimate of β and θ obtained from steps 1 and 2 along with σ_η^2 is used to estimate γ_1 by using (4.30).

Step 4: The improved estimate of β, θ and γ_1 is used to get the improved estimate of σ_η^2 by (4.33).

These cycles of iteration continues until convergence.

Chapter 5

Concluding Remarks

Using Stochastic Volatility models to analyze time series data with non-stationary variances has been popular over the last two decades. The inferences in such models have, however, proven to be difficult. The existing GMM and QML approaches are either cumbersome or inefficient. In the thesis, we have provided a simpler MM, as well as a 'working' GQL approach, to deal with this challenging inference problem. It is demonstrated through asymptotic and simulation studies that the proposed estimation approaches are simple and efficient than the existing approaches. An algorithm is given to make these approaches user friendly.

We have further proposed a new volatility model that unlike the existing stochastic volatility models, accommodates certain dynamic relationship among the responses given that the variances of the responses are also dynamically related. We have

referred to this new model as the ODDD (observation-driven dynamic dynamic) volatility model. The regression and dynamic dependence parameters have been efficiently estimated by the GQL approach, and the SMM approach has been used to estimate the volatility parameters of the dynamic model in variances. Thus, the SMM approach, which was proposed for the inferences in the standard stochastic volatility models, is demonstrated to be useful for the wider ODDD volatility models as well.

The inferences proposed for the original as well as new (ODDD) volatility models should be useful to researchers working with economic and environmental time series data, among others. The proposed estimation methodologies are extendable to the GARCH (generalized autoregressive conditional heteroscedastic) type models considered in the literature. They will also be useful to analyze volatility models with certain continuous non-normal errors.

Appendix A

Derivation of $E[\sigma_{t-k}^2 \sigma_t^2]$ [for (2.37)]:

By using the recurrence relationship of $\ln \sigma_t^2$ from (1.2), we wrote the general form for σ_t^2 as in (2.7). That is,

$$\sigma_t^2 = \exp\left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \sum_{r=0}^{t-2} \gamma_1^r \eta_{t-r}\right). \quad (\text{A.1})$$

It then follows that the product of σ_t^2 and σ_{t-k}^2 for $\text{lag } k = (1, \dots, T-2)$ and $t = (3, \dots, T)$ can be expressed as

$$\begin{aligned} \sigma_{t-k}^2 \sigma_t^2 &= \exp\left(\gamma_1^{(t-k)-1} \ln \sigma_{10}^2 + \sum_{i=0}^{(t-k)-2} \gamma_1^i \eta_{t-k-i}\right) \exp\left(\gamma_1^{t-1} \ln \sigma_{10}^2 + \sum_{j=0}^{t-2} \gamma_1^j \eta_{t-j}\right) \\ &= \exp\left(\gamma_1^{(t-k)-1} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2\right) \exp\left(\sum_{i=0}^{(t-k)-2} \gamma_1^i \eta_{t-k-i} + \sum_{j=0}^{t-2} \gamma_1^j \eta_{t-j}\right). \end{aligned}$$

Since γ_1 and σ_{10}^2 are constant, the expectation of $\sigma_{t-k}^2 \sigma_t^2$ can be computed as

$$\begin{aligned} E[\sigma_{t-k}^2 \sigma_t^2] &= \exp\left(\gamma_1^{(t-k)-1} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2\right) \\ &\quad E\left[\exp\left(\sum_{i=0}^{(t-k)-2} \gamma_1^i \eta_{t-k-i} + \sum_{j=0}^{t-2} \gamma_1^j \eta_{t-j}\right)\right], \quad (\text{A.2}) \end{aligned}$$

where η_t is random variable follows normal with mean zero and variance σ_η^2 . Since

$$\begin{aligned} \sum_{i=0}^{(t-k)-2} \gamma_1^i \eta_{t-k-i} &= \gamma_1^{(t-k)-2} \eta_2 + \gamma_1^{(t-k)-3} \eta_3 + \gamma_1^{(t-k)-4} \eta_4 + \dots \\ &\quad + \gamma_1 \eta_{(t-k)-1} + \eta_{(t-k)} \end{aligned}$$

and

$$\begin{aligned} \sum_{j=0}^{t-2} \gamma_1^j \eta_{t-j} &= \gamma_1^{t-2} \eta_2 + \gamma_1^{t-3} \eta_3 + \gamma_1^{t-4} \eta_4 + \dots + \dots \\ &\quad + \gamma_1^{k+1} \eta_{t-k-1} + \gamma_1^k \eta_{t-k} + \gamma_1^{k-1} \eta_{t-k+1} + \dots + \gamma_1 \eta_{t-1} + \eta_t, \end{aligned}$$

we obtain:

$$\begin{aligned} \sum_{v=0}^{(t-k)-2} \gamma_1^v \eta_{t-k-v} + \sum_{j=0}^{t-2} \gamma_1^j \eta_{t-j} &= \left[(1 + \gamma_1^k) \gamma_1^{(t-k)-2} \eta_2 + \dots + (1 + \gamma_1^k) \gamma_1 \eta_{t-k-1} \right. \\ &\quad \left. + (1 + \gamma_1^k) \eta_{t-k} \right] + \left[\gamma_1^{k-1} \eta_{t-k+1} + \dots + \gamma_1 \eta_{t-1} + \eta_t \right] \\ &= \left[(1 + \gamma_1^k) \sum_{l=0}^{t-k-2} \gamma_1^l \eta_{t-k-l} \right] + \left[\sum_{r=0}^{k-1} \gamma_1^r \eta_{t-r} \right], \end{aligned}$$

and hence

$$\exp\left\{ \sum_{v=0}^{(t-k)-2} \gamma_1^v \eta_{t-k-v} + \sum_{j=0}^{t-2} \gamma_1^j \eta_{t-j} \right\} = \exp\left(\left[(1 + \gamma_1^k) \sum_{l=0}^{t-k-2} \gamma_1^l \eta_{t-k-l} \right] + \left[\sum_{r=0}^{k-1} \gamma_1^r \eta_{t-r} \right] \right). \quad (\text{A.3})$$

Now, by using the assumption that $\eta_t \stackrel{iid}{\sim} N(0, \sigma_\eta^2)$, and by using normal moment generating function $E(e^{\eta}) = \exp(\sigma_\eta^2/2)$, we can compute the expectation in the second part in (A.2), as

$$E\left[\exp\left\{ \sum_{v=0}^{(t-k)-2} \gamma_1^v \eta_{t-k-v} + \sum_{j=0}^{t-2} \gamma_1^j \eta_{t-j} \right\} \right] = \exp\left[\frac{\sigma_\eta^2}{2} \left((1 + \gamma_1^k)^2 \sum_{l=0}^{t-k-2} \gamma_1^{2l} + \sum_{r=0}^{k-1} \gamma_1^{2r} \right) \right]. \quad (\text{A.4})$$

Now by using this expression from (A.4) into (A.2), we obtain

$$E_M[\sigma_{t-k}^2 \sigma_t^2] = \exp \left[\gamma_1^{(t-k)-1} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2} \left((1 + \gamma_1^k)^2 \sum_{l=0}^{t-k-2} \gamma_1^{2l} + \sum_{r=0}^{k-1} \gamma_1^{2r} \right) \right]. \quad (\text{A.5})$$

When $k = 1$ and $t=3, \dots, T$, the formula in (A.5) reduces to

$$E_M[\sigma_{t-1}^2 \sigma_t^2] = \exp \left[\gamma_1^{t-2} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2 + \frac{\sigma_\eta^2}{2} \left((1 + \gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1 \right) \right]. \quad (\text{A.6})$$

Derivation of $E_M[\sigma_{t-k}^4 \sigma_t^4]$ [for (2.50)]:

Note that the expression for σ_t^4 is given in (2.35). By using this formula from (2.35), after some algebra, we obtain

$$E_M[\sigma_{t-k}^4 \sigma_t^4] = \exp \left[2 \left(\gamma_1^{(t-k)-1} \ln \sigma_{10}^2 + \gamma_1^{t-1} \ln \sigma_{10}^2 \right) \right] E \left[\exp \left(2 \left\{ \sum_{v=0}^{(t-k)-2} \gamma_1^v \eta_{t-k-v} + \sum_{j=0}^{t-2} \gamma_1^j \eta_{t-j} \right\} \right) \right], \quad (\text{A.7})$$

where the second part in (A.7) is given by (A.4). After further algebra, we write

$$E[\sigma_{t-k}^4 \sigma_t^4] = \exp \left[2 \left(\left\{ \gamma_1^{(t-k)-1} + \gamma_1^{t-1} \right\} \ln \sigma_{10}^2 \right) + 2 \sigma_\eta^2 \left((1 + \gamma_1)^2 \sum_{l=0}^{t-k-2} \gamma_1^{2l} + \sum_{r=0}^{k-1} \gamma_1^{2r} \right) \right]. \quad (\text{A.8})$$

For lag $k=1$, and $t=2, \dots, T$, the expectation of $\sigma_{t-1}^4 \sigma_t^4$ has the expression given by

$$E[\sigma_{t-1}^4 \sigma_t^4] = \exp \left[2 \left(\left\{ \gamma_1^{t-2} + \gamma_1^{t-1} \right\} \ln \sigma_{10}^2 \right) + 2 \sigma_\eta^2 \left((1 + \gamma_1)^2 \sum_{l=0}^{t-3} \gamma_1^{2l} + 1 \right) \right]. \quad (\text{A.9})$$

Derivation of $E[\sigma_{t-k}^2 \sigma_t^4 \sigma_{t+k}^2]$ [for (2.54)]:

By using the expression for σ_t^2 from (A.1), for lag k ($=1, \dots, t-1$) and $t=(3, \dots, T)$, we may write

$$\begin{aligned} E[\sigma_{t-k}^2 \sigma_t^4 \sigma_{t+k}^2] &= \exp \left[\left(\left\{ \gamma_1^{t-k-1} + 2\gamma_1^{t-1} + \gamma_1^{t+k-1} \right\} \ln \sigma_{t0}^2 \right) \right] \\ &\quad E \left[\exp \left(\sum_{r=0}^{t-k-2} \gamma_1^r \eta_{(t-k-r)} + 2 \sum_{r=0}^{t-2} \gamma_1^r \eta_{(t-r)} + \sum_{r=0}^{t+k-2} \gamma_1^r \eta_{(t+k-r)} \right) \right]. \end{aligned} \quad (\text{A.10})$$

After some algebra, and by using the assumption that $\eta_t \stackrel{iid}{\sim} N(0, \sigma_\eta^2)$, we write

$$\begin{aligned} &E \left(\exp \left[\sum_{r=0}^{t-k-2} \gamma_1^r \eta_{(t-k-r)} + 2 \sum_{r=0}^{t-2} \gamma_1^r \eta_{(t-r)} + \sum_{r=0}^{t+k-2} \gamma_1^r \eta_{(t+k-r)} \right] \right) \\ &= E \left[\exp \left(\left\{ (1 + \gamma_1^k)^2 \sum_{i=0}^{t-k-2} \gamma_1^i \eta_{t-k-i} \right\} + \left\{ (2 + \gamma_1^k) \sum_{j=0}^{k-1} \gamma_1^j \eta_{t-j} \right\} + \left\{ \sum_{l=0}^{k-1} \gamma_1^l \eta_{t+k-l} \right\} \right) \right] \\ &= \exp \left[\frac{\sigma_\eta^2}{2} \left((1 + \gamma_1^k)^4 \sum_{i=0}^{t-k-2} \gamma_1^{2i} + (2 + \gamma_1^k)^2 \sum_{j=0}^{k-1} \gamma_1^{2j} + \sum_{l=0}^{k-1} \gamma_1^{2l} \right) \right]. \end{aligned} \quad (\text{A.11})$$

By using this formula from (A.11) into (A.10), we obtain

$$\begin{aligned} E[\sigma_{t-k}^2 \sigma_t^4 \sigma_{t+k}^2] &= \exp \left[\left(\gamma_1^{t-k-1} + 2\gamma_1^{t-1} + \gamma_1^{t+k-1} \right) \ln \sigma_{t0}^2 \right. \\ &\quad \left. + \left(\frac{\sigma_\eta^2}{2} \left\{ (1 + \gamma_1^k)^4 \sum_{i=0}^{t-k-2} \gamma_1^{2i} + (2 + \gamma_1^k)^2 \sum_{j=0}^{k-1} \gamma_1^{2j} + \sum_{l=0}^{k-1} \gamma_1^{2l} \right\} \right) \right]. \end{aligned} \quad (\text{A.12})$$

When lag $k=1$ and $t=3, \dots, T$, the formula in (A.12) reduces to

$$\begin{aligned} E[\sigma_{t-1}^2 \sigma_t^4 \sigma_{t+1}^2] &= \exp \left[\left(\gamma_1^{t-2} + 2\gamma_1^{t-1} + \gamma_1^t \right) \ln \sigma_{t0}^2 \right. \\ &\quad \left. + \left(\frac{\sigma_\eta^2}{2} \left\{ (1 + \gamma_1)^4 \sum_{i=0}^{t-3} \gamma_1^{2i} + (2 + \gamma_1)^2 + 1 \right\} \right) \right]. \end{aligned} \quad (\text{A.13})$$

Appendix B

First Order Derivatives of the Covariance Matrix \mathbf{V} w.r.t γ_1 and σ_η^2 [for (2.95)]:

For $\mathbf{V} = (v_{t-k,t})$ given in (2.85), the derivatives of the elements of this \mathbf{V} matrix w.r.t γ_1 can be computed by

$$\frac{\partial v_{t-k,t}}{\partial \gamma_1} = \begin{cases} 0 & \text{for } k=0 \text{ } t=1 \\ 0 & \text{for } t-k=1, t=2, \dots, T \\ \sigma_\eta^2 \sum_{i=2}^t 2(t-i) \gamma_1^{2(t-i)-1} & \text{for } k=0, t=2, \dots, T \\ \sigma_\eta^2 \sum_{i=2}^{t-k} (2(t-i)-k) \gamma_1^{2(t-i)-k-1} & \text{for } k=1, \dots, t-2 \text{ and } t=3, \dots, T. \end{cases} \quad (\text{B.1})$$

Similarly, the derivatives of the elements of the w.r.t σ_η^2 have the formulas

$$\frac{\partial v_{t-k,t}}{\partial \sigma_{\eta}^2} = \begin{cases} 0 & \text{for } k=0 \quad t=1 \\ 0 & \text{for } t-k=1, t=2, \dots, T \\ \sum_{i=2}^t \gamma_1^{2(t-i)} & \text{for } k=0, t=2, \dots, T \\ \sum_{i=2}^{t-k} \gamma_1^{2(t-i)-k} & \text{for } k=1, \dots, t-2 \text{ and } t=3, \dots, T. \end{cases} \quad (\text{B.2})$$

Second Order Derivatives of the Covariance Matrix \mathbf{V} w.r.t γ_1 and σ_{η}^2 [for (2.95)]:

For $\mathbf{V} = (v_{t-k,t})$ given in (2.85), the derivatives of the elements of this \mathbf{V} matrix w.r.t γ_1 can be computed by

$$\frac{\partial^2 v_{t-k,t}}{\partial \gamma_1^2} = \begin{cases} 0 & \text{for } k=0 \quad t=1 \\ 0 & \text{for } t-k=1, t=2, \dots, T \\ \sigma_{\eta}^2 \sum_{i=2}^t 2(t-i)(2(t-i)-1)\gamma_1^{2(t-i)-2} & \text{for } k=0, t=2, \dots, T \\ \sigma_{\eta}^2 \sum_{i=2}^{t-k} (2(t-i)-k)(2(t-i)-k-1)\gamma_1^{2(t-i)-k-2} & \text{for } k=1, \dots, t-2 \text{ and } t=3, \dots, T. \end{cases} \quad (\text{B.3})$$

Similarly, the derivatives of the elements of the w.r.t σ_{η}^2 have the formulas

$$\frac{\partial^2 v_{t-k,t}}{\partial (\sigma_{\eta}^2)^2} = 0 \quad k=1, \dots, t-1, \quad t=1, \dots, T. \quad (\text{B.4})$$

Second order derivatives of the quasi likelihood (QL) [for (2.94)] and their Expectation:

The computation for the second order derivatives of the QL is straightforward but lengthy. We present these derivatives below in brief.

Derivatives with respect to γ_1 and their expectation

The first order derivative of $\log L_Q^*$ w.r.t γ_1 is given in (2.92). Now, the second order derivative of (2.92) w.r.t γ_1 has the expression given by

$$\begin{aligned} \frac{\partial^2 \log L_Q^*}{\partial \gamma_1^2} &= -\frac{1}{2} \frac{\partial [\text{trace}(V^{-1} \frac{\partial V}{\partial \gamma_1})]}{\partial \gamma_1} + \frac{\partial d}{\partial \gamma_1} V^{-1} (Z - m) \\ &\quad + d \frac{\partial V^{-1}}{\partial \gamma_1} (Z - m) - d' V^{-1} d \\ &\quad + \frac{1}{2} d' \frac{\partial V^{-1}}{\partial \gamma_1} (Z - m) + \frac{1}{2} (Z - m)' \frac{\partial V^{-1}}{\partial \gamma_1} d - \\ &\quad \frac{1}{2} (Z - m)' \frac{\partial^2 V^{-1}}{\partial \gamma_1^2} (Z - m). \end{aligned} \quad (B.5)$$

Next, by taking expectation over (B.5), we obtain

$$E \left[\frac{\partial^2 \log L_Q^*}{\partial \gamma_1^2} \right] = -\frac{1}{2} \frac{\partial [\text{trace}(V^{-1} \frac{\partial V}{\partial \gamma_1})]}{\partial \gamma_1} - d' V^{-1} d - \frac{1}{2} \text{trace} \left[\frac{\partial^2 V^{-1}}{\partial \gamma_1^2} V \right],$$

where

$$\frac{\partial [\text{trace}(V^{-1} \frac{\partial V}{\partial \gamma_1})]}{\partial \gamma_1} = \text{trace} \left[\frac{\partial V^{-1}}{\partial \gamma_1} \frac{\partial V}{\partial \gamma_1} + V^{-1} \frac{\partial^2 V}{\partial \gamma_1^2} \right]$$

with

$$\frac{\partial V^{-1}}{\partial \gamma_1} = -V^{-1} \frac{\partial V}{\partial \gamma_1} V^{-1}$$

and

$$\frac{\partial^2 V^{-1}}{\partial \gamma_1^2} = -\frac{\partial V^{-1}}{\partial \gamma_1} \frac{\partial V}{\partial \gamma_1} V^{-1} - V^{-1} \frac{\partial^2 V}{\partial \gamma_1^2} V^{-1} - V^{-1} \frac{\partial V}{\partial \gamma_1} \frac{\partial V^{-1}}{\partial \gamma_1},$$

with the formulas for the derivatives of $\frac{\partial V}{\partial \gamma_1}$, as given in (B.1).

Derivatives with respect to σ_η^2 and their expectation

Similarly, the first order derivative of $\log L_Q^*$ w.r.t σ_η^2 is given in (2.93). Next, the

second order derivative of (2.93) w.r.t σ_η^2 has the expression given by

$$\frac{\partial^2 \log L}{\partial (\sigma_\eta^2)^2} = -\frac{1}{2} \frac{\partial [\text{trace}(V^{-1} \frac{\partial V}{\partial \sigma_\eta^2})]}{\partial \sigma_\eta^2} - \frac{1}{2} (\mathbf{Z} - \mathbf{m})' \frac{\partial (\frac{\partial V^{-1}}{\partial \sigma_\eta^2})}{\partial \sigma_\eta^2} (\mathbf{Z} - \mathbf{m}). \quad (\text{B.6})$$

Now, by taking expectation over (B.6), we obtain

$$E \left[\frac{\partial^2 \log L}{\partial (\sigma_\eta^2)^2} \right] = -\frac{1}{2} \frac{\partial [\text{trace}(V^{-1} \frac{\partial V}{\partial \sigma_\eta^2})]}{\partial \sigma_\eta^2} - \frac{1}{2} \text{trace} \left[\frac{\partial (\frac{\partial V^{-1}}{\partial \sigma_\eta^2})}{\partial \sigma_\eta^2} V \right] \quad (\text{B.7})$$

where

$$\begin{aligned} \frac{\partial [\text{trace}(V^{-1} \frac{\partial V}{\partial \sigma_\eta^2})]}{\partial \sigma_\eta^2} &= \text{trace} \frac{\partial V^{-1}}{\partial \sigma_\eta^2} \cdot \frac{\partial V}{\partial \sigma_\eta^2} + \text{trace} V^{-1} \frac{\partial^2 V}{\partial \sigma_\eta^2 \partial \sigma_\eta^2}, \\ \frac{\partial (\frac{\partial V^{-1}}{\partial \sigma_\eta^2})}{\partial \sigma_\eta^2} &= \frac{\partial (-V^{-1} \frac{\partial V}{\partial \sigma_\eta^2} V^{-1})}{\partial \sigma_\eta^2} \\ &= - \left[\frac{\partial V^{-1}}{\partial \sigma_\eta^2} \frac{\partial V}{\partial \sigma_\eta^2} V^{-1} + V^{-1} \frac{\partial^2 V}{\partial (\sigma_\eta^2)^2} V^{-1} + V^{-1} \frac{\partial V}{\partial \sigma_\eta^2} \frac{\partial V^{-1}}{\partial \sigma_\eta^2} \right], \end{aligned}$$

and $\frac{\partial V^{-1}}{\partial \sigma_\eta^2} = -V^{-1} \frac{\partial V}{\partial \sigma_\eta^2} V^{-1}$, with the formulas for the derivatives of $\frac{\partial V}{\partial \sigma_\eta^2}$ given in (B.2).

Derivatives with respect to γ_1 & σ_η^2 and their expectation

The first order derivative of QL w.r.t γ_1 is given in (2.92). Now taking the derivative over (2.92) w.r.t σ_η^2 has the expression given by

$$\begin{aligned} \frac{\partial^2 \log L}{\partial \gamma_1 \partial \sigma_\eta^2} &= -\frac{1}{2} \frac{\partial [\text{trace} V^{-1} \frac{\partial V}{\partial \gamma_1}]}{\partial \sigma_\eta^2} + d' \frac{\partial V^{-1}}{\partial \sigma_\eta^2} (\mathbf{Z} - \mathbf{m}) - \\ &\quad \frac{1}{2} (\mathbf{Z} - \mathbf{m})' \frac{\partial [\frac{\partial V^{-1}}{\partial \gamma_1}]}{\partial \sigma_\eta^2} (\mathbf{Z} - \mathbf{m}). \end{aligned} \quad (\text{B.8})$$

Now, the expectation over (B.8), yields

$$E \left[\frac{\partial^2 \log L}{\partial \gamma_1 \partial \sigma_\eta^2} \right] = -\frac{1}{2} \frac{\partial [\text{trace} V^{-1} \frac{\partial V}{\partial \gamma_1}]}{\partial \sigma_\eta^2} - \frac{1}{2} \text{trace} \left[\frac{\partial [\frac{\partial V^{-1}}{\partial \gamma_1}]}{\partial \sigma_\eta^2} V \right]$$

with

$$\frac{\partial V^{-1}}{\partial \sigma_\eta^2} = -V^{-1} \frac{\partial V}{\partial \sigma_\eta^2} V^{-1} \quad (\text{B.9})$$

and

$$\begin{aligned} \frac{\partial [\frac{\partial V^{-1}}{\partial \gamma_1}]}{\partial \sigma_\eta^2} &= \frac{\partial [-V^{-1} \frac{\partial V}{\partial \gamma_1} V^{-1}]}{\partial \sigma_\eta^2} \\ &= - \left[\frac{\partial V^{-1}}{\partial \sigma_\eta^2} \frac{\partial V}{\partial \gamma_1} V^{-1} + V^{-1} \frac{\partial^2 V}{\partial \sigma_\eta^2 \partial \gamma_1} V^{-1} + V^{-1} \frac{\partial V}{\partial \gamma_1} \frac{\partial V^{-1}}{\partial \sigma_\eta^2} \right]. \end{aligned} \quad (\text{B.10})$$

Appendix C

Proof of Lemma 4.1.1

Since $E[\epsilon_t] = 0$ for $t = 1, \dots, T$, one obtains $E(Y_1|\sigma_1) = x_1' \beta$, and $E(Y_t|\sigma_1, \dots, \sigma_t) = x_t' \beta$, for $t = 2, \dots, T$, by using the dynamic model (4.3). Thus,

$$E(Y_t|\sigma_1, \dots, \sigma_t) = x_t' \beta, \text{ for all } t = 1, \dots, T. \quad (\text{C.1})$$

To compute the conditional covariance, we follow the ODDD volatility model (4.3)-(4.4), first express $y_t - x_t' \beta$ as

$$\begin{aligned} (y_t - x_t' \beta) &= \theta(y_{t-1} - x_{t-1}' \beta) + \sigma_t \epsilon_t \\ &= \theta \left[\theta(y_{t-2} - x_{t-2}' \beta) + \sigma_{t-1} \epsilon_{t-1} \right] + \sigma_t \epsilon_t \\ &= \theta^2 \left[(y_{t-2} - x_{t-2}' \beta) + \theta \sigma_{t-1} \epsilon_{t-1} \right] + \sigma_t \epsilon_t \\ &= \theta^2 \left[\theta(y_{t-3} - x_{t-3}' \beta) + \sigma_{t-2} \epsilon_{t-2} \right] + \theta \sigma_{t-1} \epsilon_{t-1} + \sigma_t \epsilon_t \\ &= \theta^3 (y_{t-3} - x_{t-3}' \beta) + \theta^2 \sigma_{t-2} \epsilon_{t-2} + \theta \sigma_{t-1} \epsilon_{t-1} + \sigma_t \epsilon_t \\ &= . \end{aligned}$$

$$\begin{aligned}
&= \dots \\
&= \dots \\
&= \sigma_t \epsilon_t + \theta \sigma_{t-1} \epsilon_{t-1} + \theta^2 \sigma_{t-2} \epsilon_{t-2} \\
&\quad + \dots + \theta^{t-3} \sigma_3 \epsilon_3 + \theta^{t-2} \sigma_2 \epsilon_2 + \theta^{t-1} \sigma_1 \epsilon_1 \\
&= \sum_{j=1}^t \theta^{(t-j)} \sigma_j \epsilon_j
\end{aligned} \tag{C.2}$$

Next by using the distributional assumption for $\{\epsilon_j; j = 1, \dots, t\}$, that is, by using $E(\epsilon_j^2) = 1$ for $j = 1, \dots, t$, and $Cov(\epsilon_j, \epsilon_k) = 0$ for $j \neq k = 1, \dots, t$, we obtain the conditional covariance for $u < t$, $u = 1, \dots, T$,

$$\begin{aligned}
Cov[Y_u, Y_t | \sigma_t, \dots, \sigma_1] &= E[(Y_u - X'_u \beta)(Y_t - X'_t \beta) | \sigma_1, \dots, \sigma_1] \\
&= E \left[\sum_{j=1}^u \theta^{u-j} \sigma_j \epsilon_j \left\{ \sum_{j=1}^t \theta^{t-j} \sigma_j \epsilon_j \right. \right. \\
&\quad \left. \left. + \sum_{k=u+1}^t \theta^{t-k} \sigma_k \epsilon_k \right\} | \sigma_1, \dots, \sigma_t \right] \\
&= \sum_{j=1}^u \theta^{t+u-2j} \sigma_j^2,
\end{aligned} \tag{C.3}$$

as in (4.7).

Note that the conditional variance $Var(Y_t | \sigma_1, \dots, \sigma_t)$ follows, from (C.3) for $u = t$.

Compute the expectation of $\tilde{\varphi}_{t-1,t} = E[(y_{t-1} - x'_{t-1}\beta)^2(y_t - x'_t\beta)^2]$

By using the expression given in (C.2), write for $t=2, \dots, T$

$$\begin{aligned}
S_2^* &= (Y_{t-1} - \mu_{t-1})^2 (Y_t - \mu_t)^2 = \left(\sum_{i=1}^{t-1} \theta^{t-1-i} \sigma_i \epsilon_i \right)^2 \left(\sum_{s=1}^t \theta^{t-s} \sigma_s \epsilon_s \right)^2 \\
&= \left(\sum_{i=1}^{t-1} \theta^{t-1-i} \sigma_i \epsilon_i \right)^2 \left(\sum_{s=1}^{t-1} \theta^{t-1-s} \sigma_s \epsilon_s + \sigma_t \epsilon_t \right)^2
\end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{i=1}^{t-1} \theta^{t-1-i} \sigma_i \epsilon_i \right)^2 \left[\left(\sum_{s=1}^{t-1} \theta^{t-1-s} \sigma_s \epsilon_s \right)^2 \right. \\
&\quad \left. + 2 \left(\sum_{s=1}^{t-1} \theta^{t-1-s} \sigma_s \epsilon_s \right) \sigma_t \epsilon_t + \sigma_t^2 \epsilon_t^2 \right] \\
&= \left(\sum_{i=1}^{t-1} \theta^{t-1-i} \sigma_i \epsilon_i \right)^4 + \left(\sum_{i=1}^{t-1} \theta^{t-1-i} \sigma_i \epsilon_i \right)^2 \sigma_t^2 \epsilon_t^2 \\
&\quad + 2 \left(\sum_{i=1}^{t-1} \theta^{t-1-i} \sigma_i \epsilon_i \right)^2 \left(\sum_{s=1}^{t-1} \theta^{t-1-s} \sigma_s \epsilon_s \right) \\
&= S_{21}^* + S_{22}^* + S_{23}^* \tag{C.4}
\end{aligned}$$

The expectation of S_2^* , i.e. $\bar{\varphi}_{t-1,t} = E[(y_{t-1} - x'_{t-1}\beta)^2(y_t - x'_t\beta)^2] = E[S_2^*]$

$$\begin{aligned}
\bar{\varphi}_{t-1,t} &= EE[S_2^* | \sigma_t^2, \dots, \sigma_1^2] \\
&= EE[S_{21}^* | \sigma_t^2, \dots, \sigma_1^2] + EE[S_{22}^* | \sigma_t^2, \dots, \sigma_1^2] - E[S_{23}^*] = 0 \text{ because } E[\epsilon_i] = 0 \\
&\tag{C.5}
\end{aligned}$$

Next consider,

$$\begin{aligned}
EE[S_{21}^* | \sigma_t^2, \dots, \sigma_1^2] &= EE \left[\left(\sum_{i=1}^{t-1} \theta^{t-1-i} \sigma_i \epsilon_i \right)^4 | \sigma_t^2, \dots, \sigma_1^2 \right] \\
&= EE \left[\sum_{i=1}^{t-1} \theta^{4(t-1-i)} \sigma_i^4 \epsilon_i^4 + 4 \sum_{i=1}^{t-1} \sum_{j \neq i}^{t-1} \theta^{(4t-4-3i-j)} \sigma_i^3 \sigma_j \epsilon_i^3 \epsilon_j \right. \\
&\quad + 3 \sum_{i=1}^{t-1} \sum_{j \neq i}^{t-1} \theta^{(4t-4-2i-2j)} \sigma_i^2 \sigma_j^2 \epsilon_i^2 \epsilon_j^2 \\
&\quad + 6 \sum_{i=1}^{t-1} \sum_{j \neq i}^{t-1} \sum_{k \neq i,j}^{t-1} \theta^{(4t-4-2i-j-k)} \sigma_i^2 \sigma_j \sigma_k \epsilon_i^2 \epsilon_j \epsilon_k \\
&\quad \left. + \sum_{i=1}^{t-1} \sum_{j \neq i}^{t-1} \sum_{s \neq i,j}^{t-1} \theta^{(4t-4-i-j-k-l)} \sigma_i \sigma_j \sigma_k \sigma_l \epsilon_i \epsilon_j \epsilon_k \epsilon_l | \sigma_t^2, \dots, \sigma_1^2 \right] \\
&= E \left[3 \sum_{i=1}^{t-1} \theta^{4(t-1-i)} \sigma_i^4 + \sum_{i=1}^{t-1} \sum_{j \neq i}^{t-1} \theta^{(4t-4-2i-2j)} \sigma_i^2 \sigma_j^2 \right]
\end{aligned}$$

$$= 3 \sum_{i=1}^{t-1} \theta^{4(t-1-i)} E[\sigma_i^4] + 3 \sum_{i=1}^{t-1} \sum_{i \neq j}^{t-1} \theta^{4(t-4-2i-2j)} E[\sigma_i^2 \sigma_j^2],$$

where $E[\sigma_i^4]$ is given in (2.36) and $E[\sigma_i^2 \sigma_j^2]$ is in (A.5). Similarly,

$$\begin{aligned} EE[S_{22}^2 | \sigma_1^2, \dots, \sigma_t^2] &= EE \left[\left(\sum_{i=1}^{t-1} \theta^{t-1-i} \sigma_i \epsilon_i \right)^2 \sigma_t^2 \epsilon_t^2 | \sigma_1^2, \dots, \sigma_t^2 \right] \\ &= EE \left[\left(\sum_{i=1}^{t-1} \theta^{2(t-1-i)} \sigma_i^2 \epsilon_i^2 \right. \right. \\ &\quad \left. \left. + 2 \sum_{i=1}^{t-1} \sum_{i \neq j=1}^{t-1} \theta^{(2t-2-i-j)} \sigma_i \sigma_j \epsilon_i \epsilon_j \right) \sigma_t^2 \epsilon_t^2 | \sigma_1^2, \dots, \sigma_t^2 \right] \\ &= EE \left[\left(\sum_{i=1}^{t-1} \theta^{2(t-1-i)} \sigma_i^2 \epsilon_i^2 \sigma_t^2 \epsilon_t^2 \right. \right. \\ &\quad \left. \left. + 2 \sum_{i=1}^{t-1} \sum_{i \neq j=1}^{t-1} \theta^{(2t-2-i-j)} \sigma_i \sigma_j \epsilon_i \epsilon_j \sigma_t^2 \epsilon_t^2 \right) \sigma_t^2 \epsilon_t^2 | \sigma_1^2, \dots, \sigma_t^2 \right] \\ &= \left[\sum_{i=1}^{t-1} \theta^{2(t-1-i)} E[\sigma_i^2] \right] E[\sigma_t^2] + 0 \\ &= \left[\sum_{i=1}^{t-1} \theta^{2(t-1-i)} E[\sigma_i^2] \right] E[\sigma_t^2], \end{aligned} \tag{C.6}$$

with $E[\sigma_i^2]$ for $i=2, \dots, T$ given in (2.8).

Computation of $E_{\sigma_1^2, \dots, \sigma_t^2} [\varphi_{u-1,u}^* \varphi_{t-1,t}^* + \varphi_{u-1,t-1}^* \varphi_{u,t}^* + \varphi_{u-1,t}^* \varphi_{u,t-1}^*]$

In the same manner as above one can compute this expectation.

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